Dispersive constraints on the SM flavor structure

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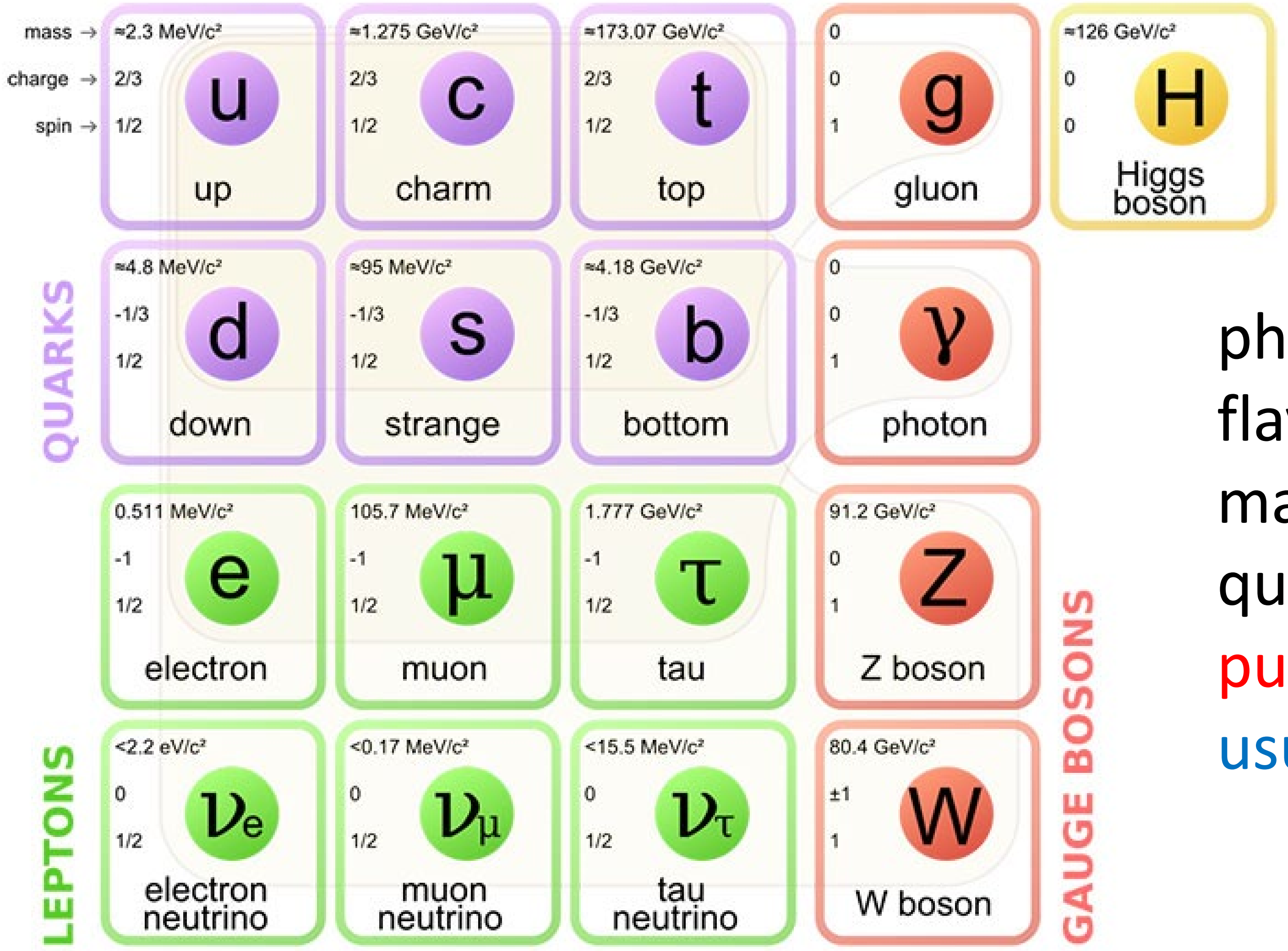
Standard Model



physicists are curious about flavor structure:

mass hierarchy (order 10E11),

quark, lepton mixing patterns,…

puzzles for decades

usually explained by new physics

Speculation

- Physical observables, being analytical, must respect dispersion relations
- Dispersion relation connects various dynamics at different scales; heavy meson lifetimes link EW and strong interactions; Higgs decays into b quark pairs link Yukawa coupling and strong interactions,…
- Numerous observables imply numerous links --- nontrivial constraints
- Perhaps SM parameters may not be completely free?
- SM flavor structure governed by dispersive constraints?
- If yes, SM flavor structure can be understood dynamically
- These studies initiated by accidental observation on D meson mixing

Mixing patterns

Why are quark and lepton mixings so different?

A simple example to demonstrate our approach

Issues about fermion mixing

see Henry's talk

- Neutrino mass ordering

$$\Delta m^2_{21} \equiv m^2_2 - m^2_1 = (7.55^{+0.20}_{-0.16}) \times 10^{-5}\ \mathrm{eV}^2 \qquad \Delta m^2_{32} \equiv m^2_3 - m^2_2 = (2.424 \pm 0.03) \times 10^{-3}\ \mathrm{eV}^2$$

but normal ordering or inverted ordering?

- Why small mixing in quark sector, but large mixing in lepton sector?

CKM: $\theta_{12} = 13.04° \pm 0.05°$, $\theta_{13} = 0.201° \pm 0.011°$, $\theta_{23} = 2.38° \pm 0.06°$

Pontecorvo–Maki–Nakagawa–Sakata: $\boldsymbol{\theta_{12} = 33.41^{\circ}\,{}^{+0.75^\circ}_{-0.72^\circ}}$ $\quad \theta_{13} = 8.54^{\circ}\,{}^{+0.11^\circ}_{-0.12^\circ}$

- Why lepton mixing has maximal angle $\theta_{23} \approx 45^\circ$?

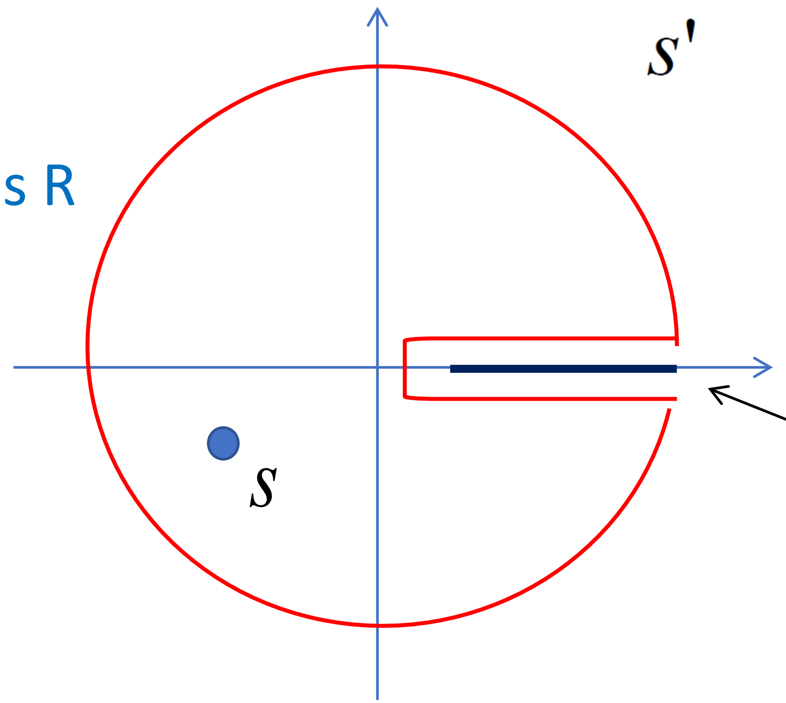
Dispersion relation

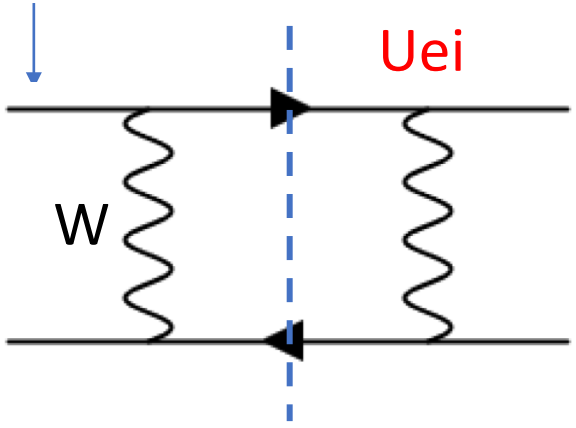
- $\mu^- e^+$-$\mu^+ e^-$ mixing amplitude $\Pi(s) \equiv M(s) - i\Gamma(s)/2$

$$M(s) = \frac{1}{2\pi}\int^R ds' \frac{\Gamma(s')}{s - s'} + \frac{1}{2\pi i}\int_{C_R} ds' \frac{\Pi(s')}{s' - s}$$

real part imaginary part

so far, it's identity for physical observable based on analyticity

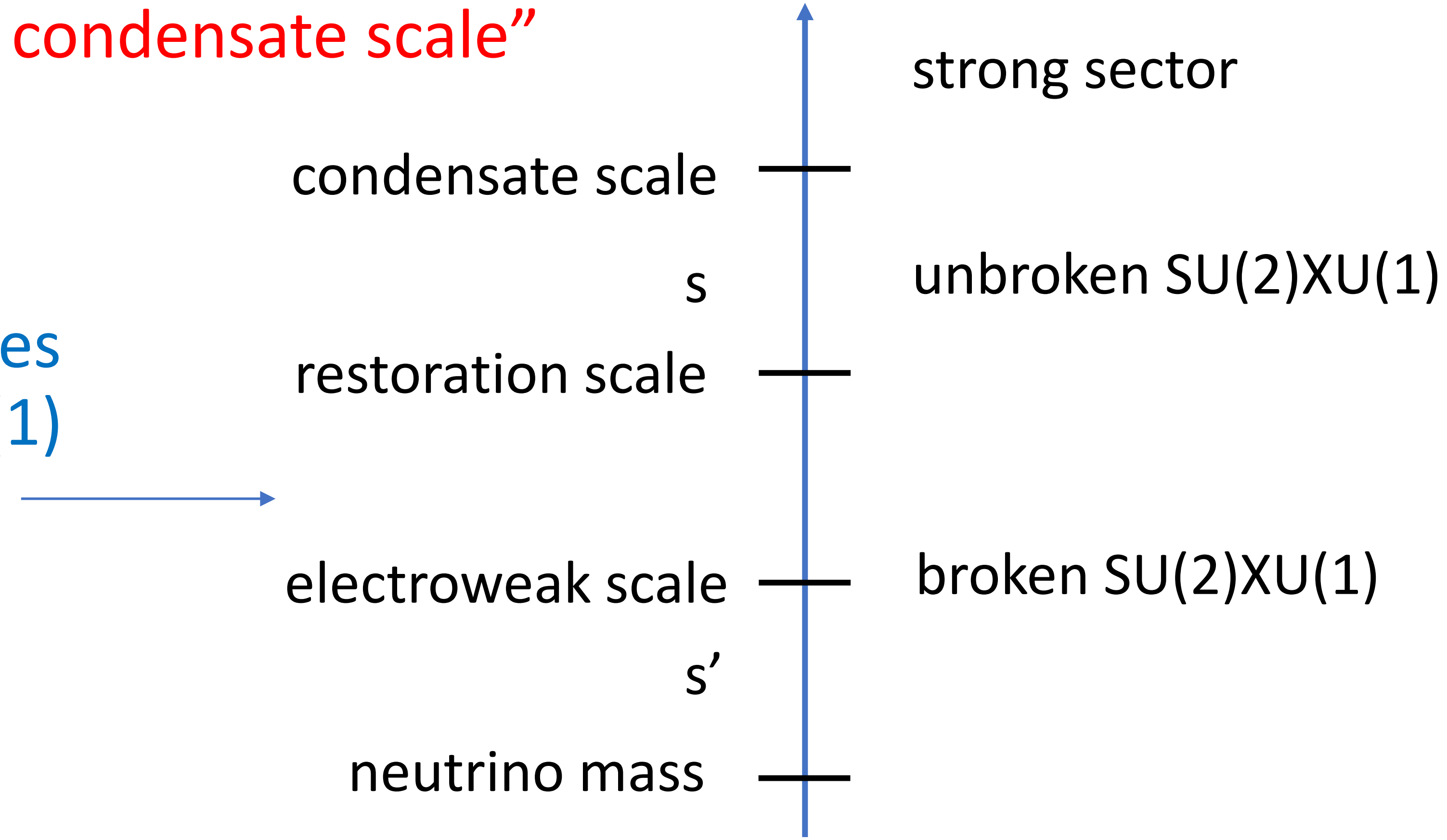




branch cut caused by Intermediate real states

What if EW symmetry restored at high energy?

- Composite Higgs model, Kaplan and Georgi, Phys. Lett. B136, 183 (1984):
- “The electroweak group is broken at a scale much smaller than the condensate scale”
- Hyperquark condensates misalign with SU(2)XU(1) vacuum owing to Yukawa couplings

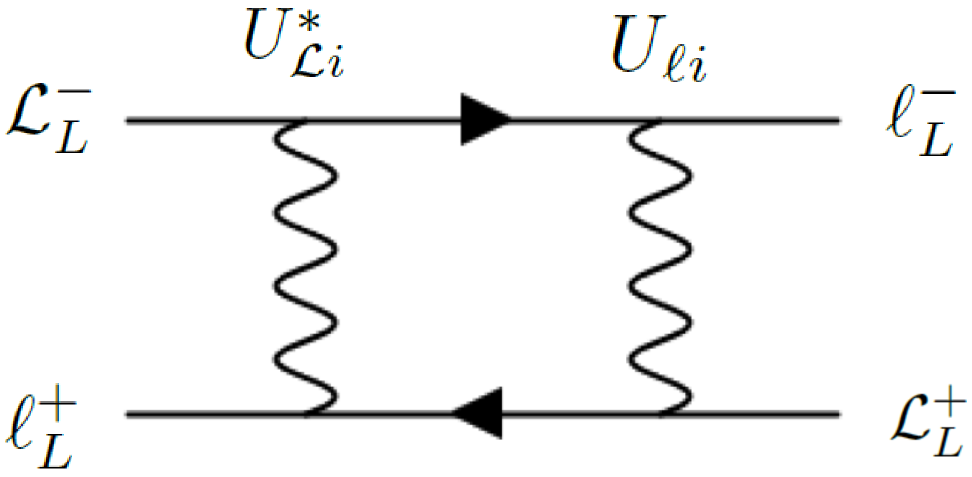


strong sector
condensate scale
unbroken SU(2)XU(1)
s
restoration scale
electroweak scale
broken SU(2)XU(1)
s’
neutrino mass

LO mixing in symmetric phase

Li, 2306.03463

- Internal particles massless
- All intermediate channels give same contribution
- Sum over all channels vanishes due to unitarity $\sum_i U^*_{\mathcal{L}i} U_{\ell i} = 0$
- Mixing phenomenon disappears!



restoration scale

$$M(s) = \frac{1}{2\pi} \int^{\overset{\downarrow}{\Lambda}} ds' \, \frac{\Gamma(s')}{s - s'} \underset{s > \Lambda}{\underline{\approx 0}}$$

EW symmetry broken at low energy;

constrains fermion masses and mixing angles

Box diagram in broken phase

- s’ can be low, so $\Gamma(s')$ depends on PMNS matrix elements and intermediate neutrino masses in broken phase.
- Box-diagram contribution from channel with two real neutrinos

Cheng 1982
Buras et al 1984

$$\Gamma(s) \propto \sum_{i,j=1}^{3} \lambda_i \lambda_j \Gamma_{ij}(s), \qquad \lambda_i = U_{\mathcal{L}i} U_{\ell i}$$

$$\begin{aligned}\Gamma_{ij}(s) \;&=\; \frac{1}{s^2} \frac{\sqrt{s^2 - 2s(m_i^2 + m_j^2) + (m_i^2 - m_j^2)^2}}{(m_W^2 - m_i^2)(m_W^2 - m_j^2)} \\ &\times \left\{ \left(m_W^4 + \frac{m_i^2 m_j^2}{4} \right) [2s^2 - 4s(m_i^2 + m_j^2) + 2(m_i^2 - m_j^2)^2] + 3m_W^2 s(m_i^2 + m_j^2)(m_i^2 + m_j^2 - s) \right\}\end{aligned}$$

Constraints

- How to diminish dispersive integral $\int^{\Lambda} ds' \frac{\Gamma(s')}{s-s'}$?
- Asymptotic expansion

to have finite integral

$$\boxed{\sum_{i,j} \lambda_i \lambda_j \Gamma_{ij}^{(m)} \approx 0, \quad m = 1, 0, -1}$$

$$\Gamma_{ij}(s') \approx \Gamma_{ij}^{(1)} s' + \Gamma_{ij}^{(0)} + \frac{\Gamma_{ij}^{(-1)}}{s'} + \cdots$$

$$\Gamma_{ij}^{(1)} = \frac{4m_W^4 - 6m_W^2(m_i^2 + m_j^2) + 4m_i^2 m_j^2}{2(m_W^2 - m_i^2)(m_W^2 - m_j^2)}, \quad \Rightarrow \quad \Lambda^2/s$$

$$\Gamma_{ij}^{(0)} = -\frac{3(m_i^2 + m_j^2)\left[4m_W^4 - 4m_W^2(m_i^2 + m_j^2) + m_i^2 m_j^2\right]}{2(m_W^2 - m_i^2)(m_W^2 - m_j^2)} \quad \Rightarrow \quad (m_i^2 + m_j^2)\Lambda/s$$

$$\Gamma_{ij}^{(-1)} = \frac{3(m_i^4 + m_j^4)\left[4m_W^4 - 2m_W^2(m_i^2 + m_j^2) + m_i^2 m_j^2\right]}{2(m_W^2 - m_i^2)(m_W^2 - m_j^2)}. \quad \Rightarrow \quad (m_i^4 + m_j^4)\ln \Lambda/s$$

to diminish integral

$$\int ds' \frac{\Gamma_{12}(s')}{s-s'} \approx \frac{1}{s}\sum_{i,j} \lambda_i \lambda_j g_{ij} \qquad g_{ij} \equiv \int_{t_{ij}}^{\infty} ds' \left[\Gamma_{ij}(s') - \Gamma_{ij}^{(1)} s' - \Gamma_{ij}^{(0)} - \frac{\Gamma_{ij}^{(-1)}}{s'}\right]$$

$$\boxed{\sum_{i,j} \lambda_i \lambda_j g_{ij} \approx 0}$$

These four conditions constrain neutrino masses and mixing angles!

Test quark mixing first---constrain quark masses and CKM matrix elements

for D mixing $\lambda_i \equiv V_{ci}^{*} V_{ui} \qquad i, j = d, s, b$

Minimization

- Use unitarity to eliminate λ_b and to rewrite constraints

$$r^2 R_{dd}^{(m)} + 2r R_{ds}^{(m)} + 1 \approx 0, , \quad m = 1, 0, -1, i$$

← refer to finite integral g_{ij}

$$R_{dd}^{(m)} = \frac{\Gamma_{dd}^{(m)} - 2\Gamma_{db}^{(m)} + \Gamma_{bb}^{(m)}}{\Gamma_{ss}^{(m)} - 2\Gamma_{sb}^{(m)} + \Gamma_{bb}^{(m)}}, \quad R_{ds}^{(m)} = \frac{\Gamma_{ds}^{(m)} - \Gamma_{db}^{(m)} - \Gamma_{sb}^{(m)} + \Gamma_{bb}^{(m)}}{\Gamma_{ss}^{(m)} - 2\Gamma_{sb}^{(m)} + \Gamma_{bb}^{(m)}} \qquad m = 1, 0, -1$$

- Expression for $m = i$ similar, but with g_{ij}
- Ratio of CKM elements

$$r = \frac{\lambda_d}{\lambda_s} = \frac{V_{cd}^* V_{ud}}{V_{cs}^* V_{us}} \equiv u + iv,$$

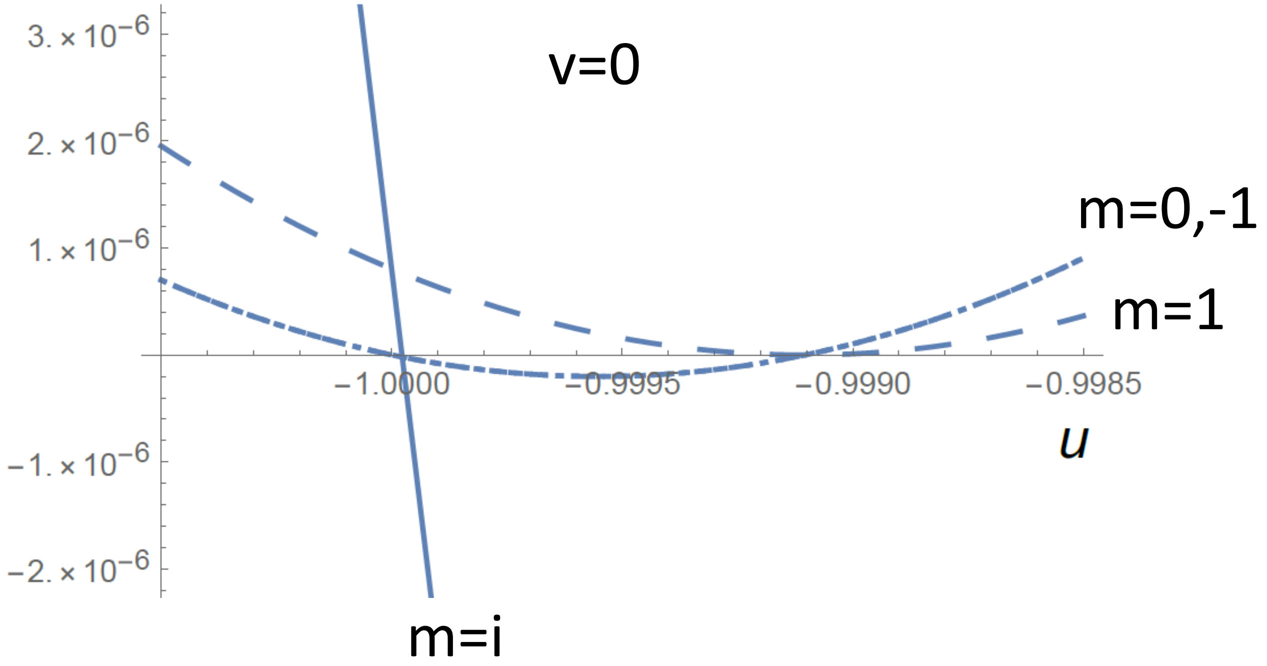
- Tune u and v to minimize the sum (real parts of constraints)

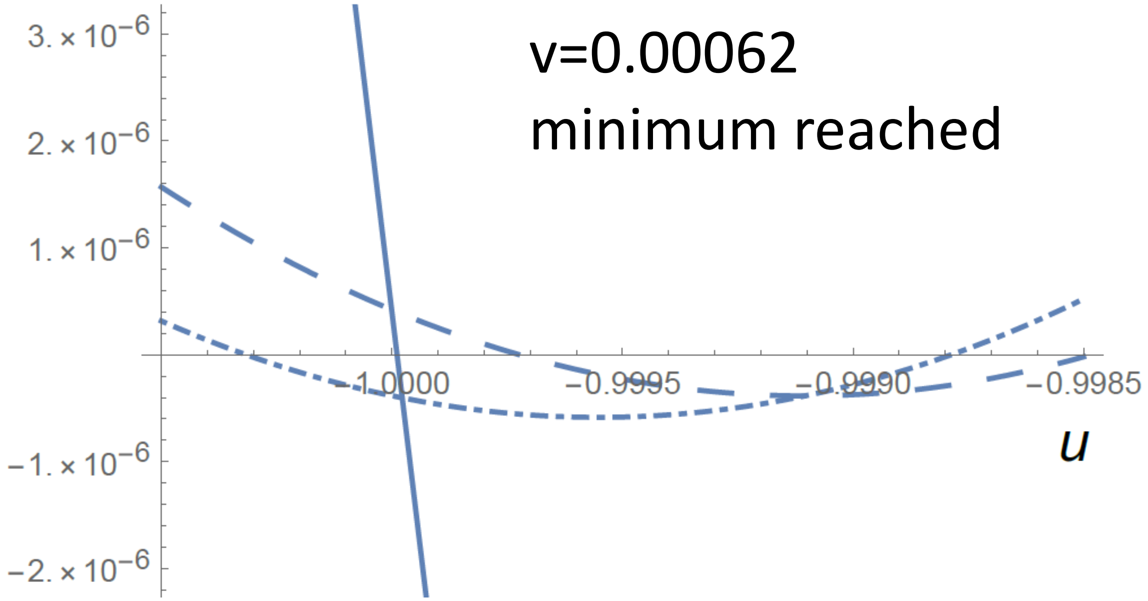
$$\sum_{m=1,-1,i} \left[(u^2 - v^2) R_{dd}^{(m)} + 2u R_{ds}^{(m)} + 1 \right]^2$$

then imaginary parts also small

Results

$$m_d = 0.005\ \text{GeV} \quad m_s = 0.12\ \text{GeV} \quad m_b = 4.0\ \text{GeV} \quad m_W = 80.377\ \text{GeV}$$





PDG

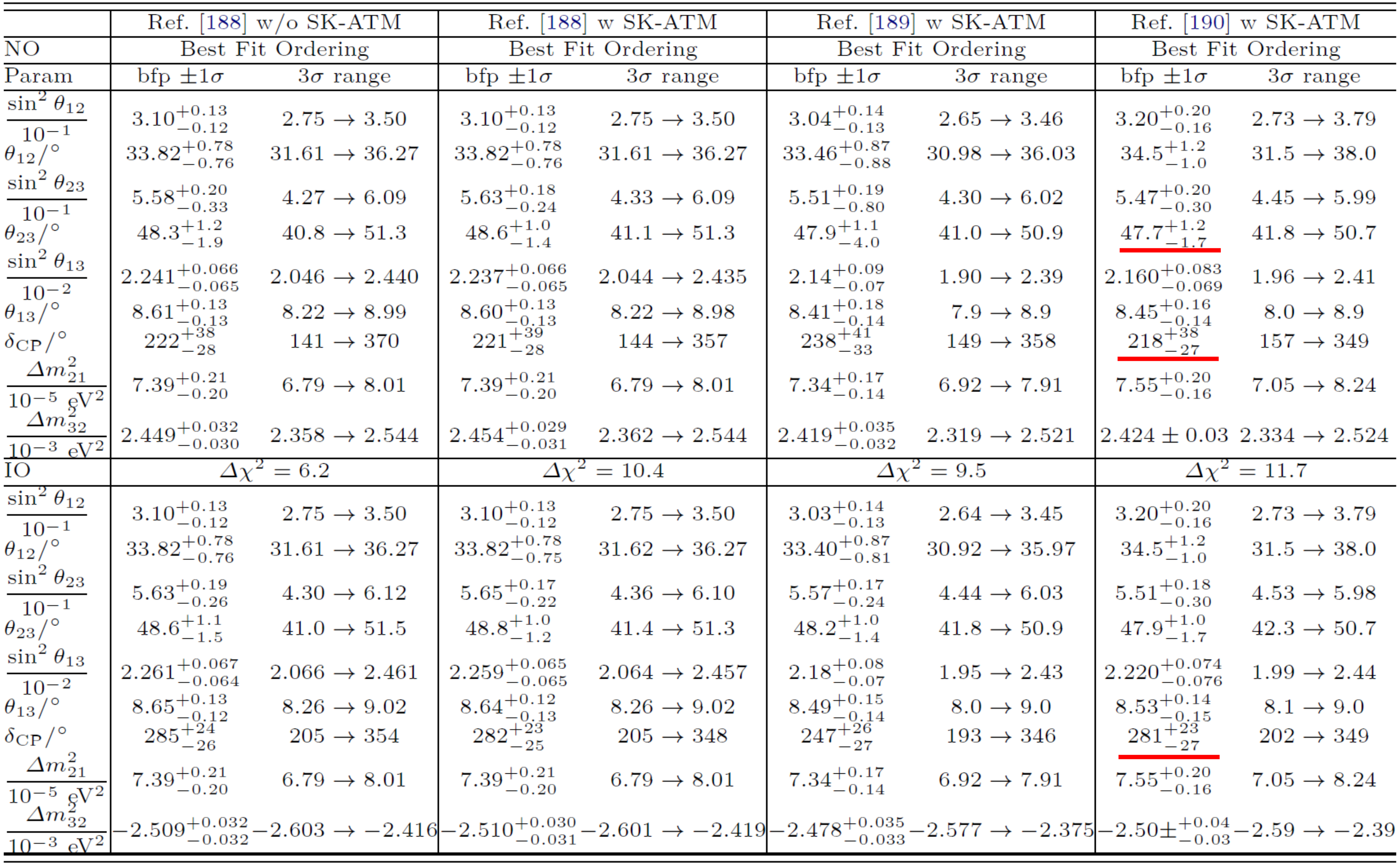
$$r = \frac{V^*_{cd}V_{ud}}{V^*_{cs}V_{us}} = -1.0 + (6.2^{+1.2}_{-1.0}) \times 10^{-4} i \qquad u = -1.00029 \pm 0.00002, \quad v = 0.00064 \pm 0.00002$$

↑ variation of ms by 0.01 GeV

they agree well; CP phase must exist

Global fits

experimental discrimination of NO, IO difficult



	Ref. [188] w/o SK-ATM		Ref. [188] w SK-ATM		Ref. [189] w SK-ATM		Ref. [190] w SK-ATM	
NO	Best Fit Ordering		Best Fit Ordering		Best Fit Ordering		Best Fit Ordering	
Param	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
$\frac{\sin^2\theta_{12}}{10^{-1}}$	$3.10^{+0.13}_{-0.12}$	$2.75 \rightarrow 3.50$	$3.10^{+0.13}_{-0.12}$	$2.75 \rightarrow 3.50$	$3.04^{+0.14}_{-0.13}$	$2.65 \rightarrow 3.46$	$3.20^{+0.20}_{-0.16}$	$2.73 \rightarrow 3.79$
$\theta_{12}/^\circ$	$33.82^{+0.78}_{-0.76}$	$31.61 \rightarrow 36.27$	$33.82^{+0.78}_{-0.76}$	$31.61 \rightarrow 36.27$	$33.46^{+0.87}_{-0.88}$	$30.98 \rightarrow 36.03$	$34.5^{+1.2}_{-1.0}$	$31.5 \rightarrow 38.0$
$\frac{\sin^2\theta_{23}}{10^{-1}}$	$5.58^{+0.20}_{-0.33}$	$4.27 \rightarrow 6.09$	$5.63^{+0.18}_{-0.24}$	$4.33 \rightarrow 6.09$	$5.51^{+0.19}_{-0.80}$	$4.30 \rightarrow 6.02$	$5.47^{+0.20}_{-0.30}$	$4.45 \rightarrow 5.99$
$\theta_{23}/^\circ$	$48.3^{+1.2}_{-1.9}$	$40.8 \rightarrow 51.3$	$48.6^{+1.0}_{-1.4}$	$41.1 \rightarrow 51.3$	$47.9^{+1.1}_{-4.0}$	$41.0 \rightarrow 50.9$	$47.7^{+1.2}_{-1.7}$	$41.8 \rightarrow 50.7$
$\frac{\sin^2\theta_{13}}{10^{-2}}$	$2.241^{+0.066}_{-0.065}$	$2.046 \rightarrow 2.440$	$2.237^{+0.066}_{-0.065}$	$2.044 \rightarrow 2.435$	$2.14^{+0.09}_{-0.07}$	$1.90 \rightarrow 2.39$	$2.160^{+0.083}_{-0.069}$	$1.96 \rightarrow 2.41$
$\theta_{13}/^\circ$	$8.61^{+0.13}_{-0.13}$	$8.22 \rightarrow 8.99$	$8.60^{+0.13}_{-0.13}$	$8.22 \rightarrow 8.98$	$8.41^{+0.18}_{-0.14}$	$7.9 \rightarrow 8.9$	$8.45^{+0.16}_{-0.14}$	$8.0 \rightarrow 8.9$
$\delta_{\rm CP}/^\circ$	222^{+38}_{-28}	$141 \rightarrow 370$	221^{+39}_{-28}	$144 \rightarrow 357$	238^{+41}_{-33}	$149 \rightarrow 358$	218^{+38}_{-27}	$157 \rightarrow 349$
$\frac{\Delta m^2_{21}}{10^{-5}\ {\rm eV}^2}$	$7.39^{+0.21}_{-0.20}$	$6.79 \rightarrow 8.01$	$7.39^{+0.21}_{-0.20}$	$6.79 \rightarrow 8.01$	$7.34^{+0.17}_{-0.14}$	$6.92 \rightarrow 7.91$	$7.55^{+0.20}_{-0.16}$	$7.05 \rightarrow 8.24$
$\frac{\Delta m^2_{32}}{10^{-3}\ {\rm eV}^2}$	$2.449^{+0.032}_{-0.030}$	$2.358 \rightarrow 2.544$	$2.454^{+0.029}_{-0.031}$	$2.362 \rightarrow 2.544$	$2.419^{+0.035}_{-0.032}$	$2.319 \rightarrow 2.521$	2.424 ± 0.03	$2.334 \rightarrow 2.524$
IO	$\Delta\chi^2 = 6.2$		$\Delta\chi^2 = 10.4$		$\Delta\chi^2 = 9.5$		$\Delta\chi^2 = 11.7$	
$\frac{\sin^2\theta_{12}}{10^{-1}}$	$3.10^{+0.13}_{-0.12}$	$2.75 \rightarrow 3.50$	$3.10^{+0.13}_{-0.12}$	$2.75 \rightarrow 3.50$	$3.03^{+0.14}_{-0.13}$	$2.64 \rightarrow 3.45$	$3.20^{+0.20}_{-0.16}$	$2.73 \rightarrow 3.79$
$\theta_{12}/^\circ$	$33.82^{+0.78}_{-0.76}$	$31.61 \rightarrow 36.27$	$33.82^{+0.78}_{-0.75}$	$31.62 \rightarrow 36.27$	$33.40^{+0.87}_{-0.81}$	$30.92 \rightarrow 35.97$	$34.5^{+1.2}_{-1.0}$	$31.5 \rightarrow 38.0$
$\frac{\sin^2\theta_{23}}{10^{-1}}$	$5.63^{+0.19}_{-0.26}$	$4.30 \rightarrow 6.12$	$5.65^{+0.17}_{-0.22}$	$4.36 \rightarrow 6.10$	$5.57^{+0.17}_{-0.24}$	$4.44 \rightarrow 6.03$	$5.51^{+0.18}_{-0.30}$	$4.53 \rightarrow 5.98$
$\theta_{23}/^\circ$	$48.6^{+1.1}_{-1.5}$	$41.0 \rightarrow 51.5$	$48.8^{+1.0}_{-1.2}$	$41.4 \rightarrow 51.3$	$48.2^{+1.0}_{-1.4}$	$41.8 \rightarrow 50.9$	$47.9^{+1.0}_{-1.7}$	$42.3 \rightarrow 50.7$
$\frac{\sin^2\theta_{13}}{10^{-2}}$	$2.261^{+0.067}_{-0.064}$	$2.066 \rightarrow 2.461$	$2.259^{+0.065}_{-0.065}$	$2.064 \rightarrow 2.457$	$2.18^{+0.08}_{-0.07}$	$1.95 \rightarrow 2.43$	$2.220^{+0.074}_{-0.076}$	$1.99 \rightarrow 2.44$
$\theta_{13}/^\circ$	$8.65^{+0.13}_{-0.12}$	$8.26 \rightarrow 9.02$	$8.64^{+0.12}_{-0.13}$	$8.26 \rightarrow 9.02$	$8.49^{+0.15}_{-0.14}$	$8.0 \rightarrow 9.0$	$8.53^{+0.14}_{-0.15}$	$8.1 \rightarrow 9.0$
$\delta_{\rm CP}/^\circ$	285^{+24}_{-26}	$205 \rightarrow 354$	282^{+23}_{-25}	$205 \rightarrow 348$	247^{+26}_{-27}	$193 \rightarrow 346$	281^{+23}_{-27}	$202 \rightarrow 349$
$\frac{\Delta m^2_{21}}{10^{-5}\ {\rm eV}^2}$	$7.39^{+0.21}_{-0.20}$	$6.79 \rightarrow 8.01$	$7.39^{+0.21}_{-0.20}$	$6.79 \rightarrow 8.01$	$7.34^{+0.17}_{-0.14}$	$6.92 \rightarrow 7.91$	$7.55^{+0.20}_{-0.16}$	$7.05 \rightarrow 8.24$
$\frac{\Delta m^2_{32}}{10^{-3}\ {\rm eV}^2}$	$-2.509^{+0.032}_{-0.032}$	$-2.603 \rightarrow -2.416$	$-2.510^{+0.030}_{-0.031}$	$-2.601 \rightarrow -2.419$	$-2.478^{+0.035}_{-0.033}$	$-2.577 \rightarrow -2.375$	$-2.50\pm^{+0.04}_{-0.03}$	$-2.59 \rightarrow -2.39$

Chau-Keung parametrization

Pontecorvo–Maki–Nakagawa–Sakata matrix

U =

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{bmatrix}.$$

Neutrino mass orderings

- Apply to lepton $\mu^- e^+$-$\mu^+ e^-$ mixing with intermediate neutrino channels
- Normal ordering (NO) $m_1^2 = 10^{-6}\ \text{eV}^2$ (as long as it is small enough)

$$\Delta m_{21}^2 \equiv m_2^2 - m_1^2 = (7.55^{+0.20}_{-0.16}) \times 10^{-5}\ \text{eV}^2 \qquad \Delta m_{32}^2 \equiv m_3^2 - m_2^2 = (2.424 \pm 0.03) \times 10^{-3}\ \text{eV}^2$$

de Salas et al, 2018

- Predict

$$r = \frac{U_{\mu 1}^* U_{e1}}{U_{\mu 2}^* U_{e2}} \approx \underline{-1.0} - 0.02i$$

global fit

$$r = -\underline{(0.738^{+0.050}_{-0.048})} - (0.179^{+0.136}_{-0.125})i$$

- Be reminded that it is LO analysis with 3 generations
- Inverted ordering (IO) $r \approx -1.0 - O(10^{-5})i$ $\quad r = -(1.03^{+0.05}_{-0.16}) - (0.356^{+0.015}_{-0.048})i$

dramatically different

- NO and observed PMNS matrix satisfy constraint at order of magnitude

Mixing patterns

- Insert u=-1 into m=1 constraint to get analytical expression of v

$$v \approx \frac{(m_W^2 - m_b^2)(m_s^2 - m_d^2)}{(m_W^2 - m_s^2)(m_b^2 - m_d^2)} \approx \frac{m_s^2}{m_b^2}$$

- In terms of Wolfenstein parameters $v = A^2\lambda^4\eta$ Ahn et al, 2011
- Produce well-known empirical relation (Cheng, Sher 1987)

$$\lambda = V_{us} \approx (A^2\eta)^{-1/4}\sqrt{\frac{m_s}{m_b}} \approx \sqrt{\frac{m_s}{m_b}}$$

$$A \approx 0.826 \qquad \eta \approx 0.348$$

$$(A^2\eta)^{-1/4} \approx 1.43 \sim O(1)$$

Belfatto et al, 2023

- Chau-Keung parametrization $V_{us} \approx s_{12}$
- Larger mixing angles in lepton sector due to $\boxed{\frac{m_2^2}{m_3^2} \approx 3.1 \times 10^{-2} \gg \frac{m_s^2}{m_b^2} \approx 9.0 \times 10^{-4}}$
- Indeed, $\sqrt{m_s/m_b}/\sqrt{m_2/m_3} \approx s_{12}^{CKM}/s_{12}^{PNMS} \approx 0.42$

Mixing of generations 1-3

- Heavy lepton could be μ or τ , same intermediate neutrinos
- τ^-e^+-τ^+e^- and μ^-e^+-μ^+e^- satisfy same constraints?
- Magnitude of PMNS matrix elements

$$|U| = \begin{bmatrix} |U_{e1}| & |U_{e2}| & |U_{e3}| \\ |U_{\mu 1}| & |U_{\mu 2}| & |U_{\mu 3}| \\ |U_{\tau 1}| & |U_{\tau 2}| & |U_{\tau 3}| \end{bmatrix} = \begin{bmatrix} 0.803 \sim 0.845 & 0.514 \sim 0.578 & 0.142 \sim 0.155 \\ 0.233 \sim 0.505 & 0.460 \sim 0.693 & 0.630 \sim 0.779 \\ 0.262 \sim 0.525 & 0.473 \sim 0.702 & 0.610 \sim 0.762 \end{bmatrix}$$

NuFIT, 2023

These two rows are indeed similar

Maximal mixing angle θ_{23}

- Recall v has two solutions with opposite signs, so one for $\mu^- e^+$-$\mu^+ e^-$ another for $\tau^- e^+$-$\tau^+ e^-$?
- Check data

$U^*_{\tau 1} U_{e1}/(U^*_{\tau 2} U_{e2})$	$r = U^*_{\mu 1} U_{e1}/(U^*_{\mu 2} U_{e2})$	
$-(1.231^{+0.078}_{-0.186}) \underline{+ (0.204^{+0.085}_{-0.138})} i$	$r = -(0.738^{+0.050}_{-0.048}) \underline{- (0.179^{+0.136}_{-0.125})} i$	de Salas et al, 2018
$-(1.139^{+0.139}_{-0.207}) \underline{+ (0.266^{+0.050}_{-0.124})} i$	$r = -(0.801^{+0.219}_{-0.097}) \underline{- (0.265^{+0.090}_{-0.145})} i$	Capozzi et al, 2018

- Implication: $\theta_{23} \approx 45^\circ$

roughly equal

$$\frac{U^*_{\mu 1} U_{e1}}{U^*_{\mu 2} U_{e2}} = -\frac{c_{12}}{s_{12}} \frac{c_{12} s_{12}(c^2_{23} - s^2_{13} s^2_{23}) + c_{23} s_{13} s_{23} c_\delta (c^2_{12} - s^2_{12}) - c_{23} s_{13} s_{23} s_\delta i}{(c_{12} c_{23} - s_{12} s_{13} s_{23})^2 + 2 c_{12} c_{23} s_{12} s_{13} s_{23} (1 - c_\delta)}$$

roughly equal

$$\frac{U^*_{\tau 1} U_{e1}}{U^*_{\tau 2} U_{e2}} = -\frac{c_{12}}{s_{12}} \frac{c_{12} s_{12}(s^2_{23} - c^2_{23} s^2_{13}) - c_{23} s_{13} s_{23} c_\delta (c^2_{12} - s^2_{12}) + c_{23} s_{13} s_{23} s_\delta i}{(c_{12} s_{23} + c_{23} s_{12} s_{13})^2 - 2 c_{12} c_{23} s_{12} s_{13} s_{23} (1 - c_\delta)}$$

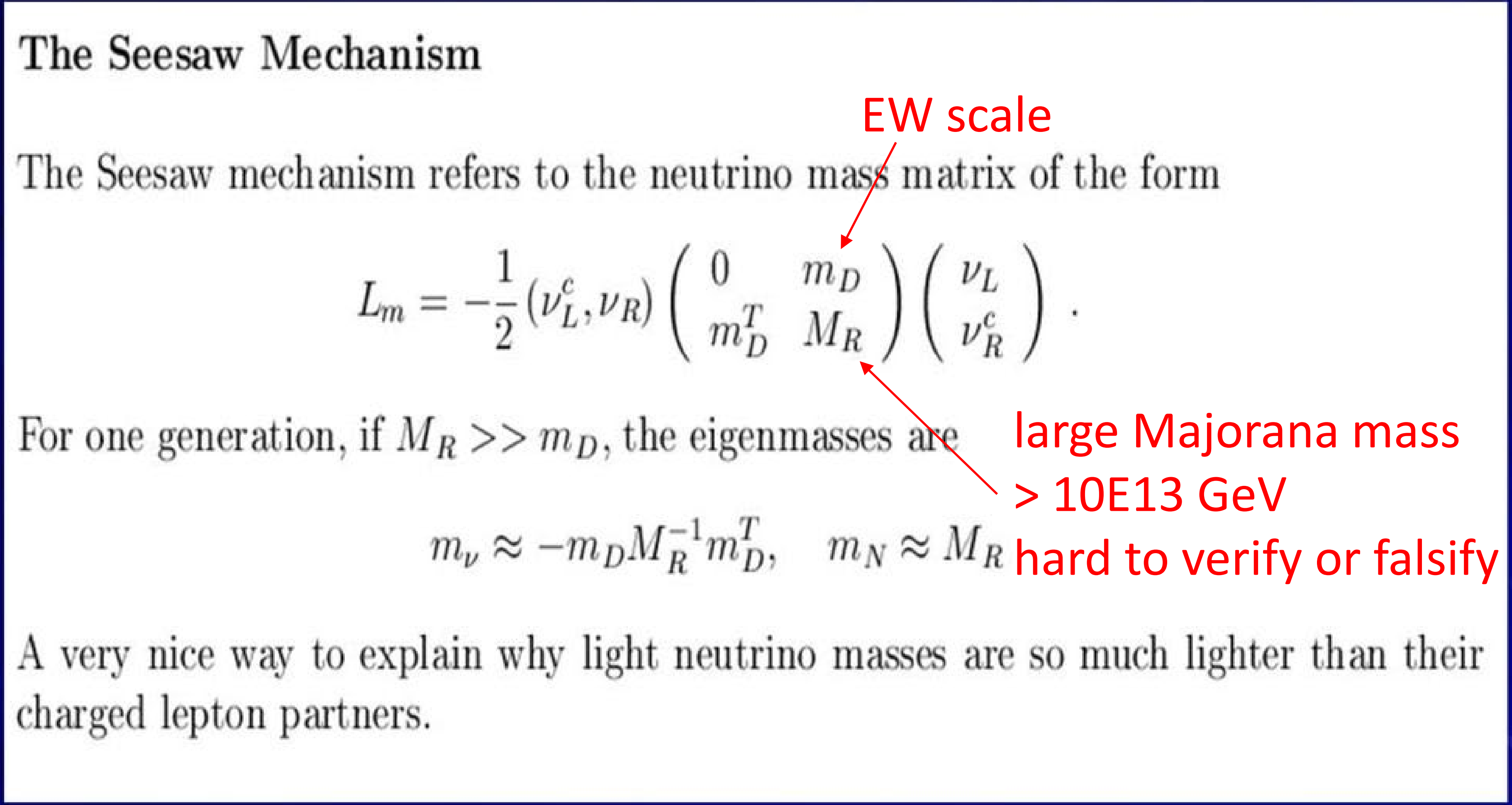
$$(c^2_{12} + s^2_{12} s^2_{13})(c^2_{23} - s^2_{23}) \approx 0$$

Neutrino mass

Why are neutrinos so light?

Concerned only mass ratios previously; how about absolute mass?

Not about seesaw mechanism



The Seesaw Mechanism

EW scale

The Seesaw mechanism refers to the neutrino mass matrix of the form

$$L_m = -\frac{1}{2}(\nu_L^c, \nu_R)\begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix}\begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix}.$$

For one generation, if $M_R >> m_D$, the eigenmasses are

$$m_\nu \approx -m_D M_R^{-1} m_D^T, \quad m_N \approx M_R$$

large Majorana mass
> 10E13 GeV
hard to verify or falsify

A very nice way to explain why light neutrino masses are so much lighter than their charged lepton partners.

Minkowski; Yanagida; Gell-Man, Ramond and Slansky…

Formalism

- Decomposition

$$\Pi(s) = \sum_{i,j=1}^{3} \lambda_i \lambda_j \Pi_{ij}(s) + \cdots \equiv \sum_{i,j=1}^{3} \lambda_i \lambda_j \left[M_{ij}(s) - \frac{i}{2}\Gamma_{ij}(s) \right] + \cdots$$

$$\lambda_i \equiv U^*_{\mathcal{L}i} U_{\ell i}$$

higher powers in $\lambda_{i,j}$

- Dispersion relation

$$M_{ij}(s) = \frac{1}{2\pi} \int^{R} ds' \frac{\Gamma_{ij}(s')}{s - s'} + \frac{1}{2\pi i} \int_{C_R} ds' \frac{\Pi_{ij}(s')}{s' - s}$$

- Unitary fermion transform as in broken phase

$$\nu_L^{(f)} = U_\nu \nu_L, \quad \nu_R^{(f)} = V_\nu \nu_R, \quad \ell_L^{(f)} = U_\ell \ell_L, \quad \ell_R^{(f)} = V_\ell \ell_R$$

Yukawa matrix elements are not all independent

- Yukawa matrices diagonalized $Y_\nu^{(d)} = U_\nu^\dagger Y_\nu V_\nu, \quad Y_\ell^{(d)} = U_\ell^\dagger Y_\ell V_\ell$
- Define PMNS matrix $U = U_\ell^\dagger U_\nu$
- Lagrangian for symmetric phase

physical mass eigenstate

$$\ell_L = U \ell'_L, \quad \nu_L = U^\dagger \nu'_L$$

assume Dirac neutrinos

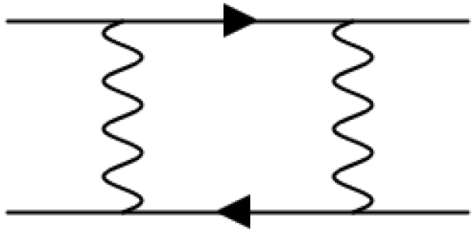
$$Y_\nu^{(d)} \bar{\nu}_L (-\bar{\phi}^0) \nu_R + Y_\ell^{(d)} \bar{\ell}_L \phi^0 \ell_R + Y_\nu^{(d)} \bar{\ell}'_L \phi^- \nu_R + Y_\ell^{(d)} \bar{\nu}'_L \phi^+ \ell_R + h.c.$$

Higgs

charged scalar

Mij in symmetric phase

- One-loop

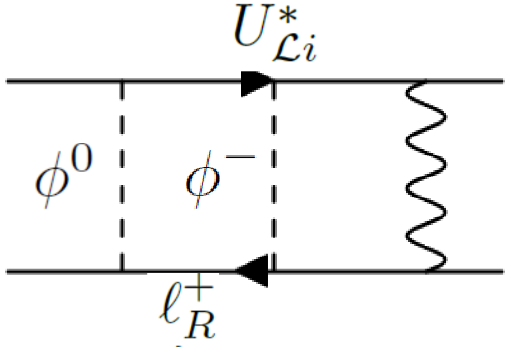


$$= -\frac{1}{16\pi^2}\left(\frac{g}{2\sqrt{2}}\right)^4 \frac{4}{s}$$

↑ weak coupling

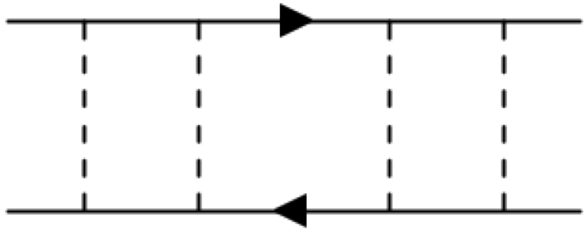
external state in broken phase; first emissions composed of neutral scalar and gauge bosons

- Two-loop

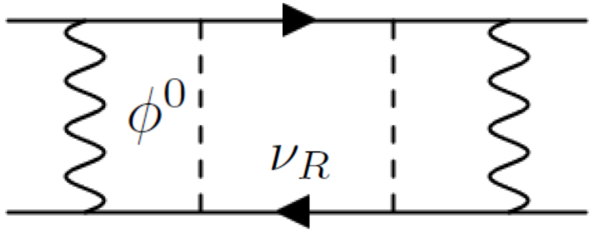


$$= -\left(\frac{1}{16\pi^2}\right)^2\left(\frac{g}{2\sqrt{2}}\right)^2 \frac{m_{\mathcal{L}}^2 m_\ell^2}{v^4}\frac{4}{s}$$

- Three-loop



$$\approx \frac{1}{64}\left(\frac{1}{16\pi^2}\right)^3 \frac{m_{\mathcal{L}}^4 m_\ell^4}{v^8}\frac{3}{s}$$



$$O\left(g^4\frac{m_i^2 m_j^2}{v^4}\right)$$

first term surviving summation over all channels

$$\sum_i U^*_{\mathcal{L}i}U_{\ell i}(Y_\nu^{(d)})^2_{ii} \neq 0$$

Γ_{ij} in broken phase

- Mij implies mixing amplitude decreases like 1/s in symmetric phase

remove

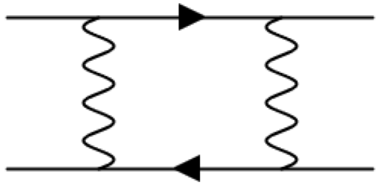
$\Lambda^2 < s' < R$

$$M_{ij}(s) = \frac{1}{2\pi}\int^{R} ds' \frac{\Gamma_{ij}(s')}{s-s'} + \frac{1}{2\pi i}\int_{C_R} ds' \frac{\Pi_{ij}(s')}{s'-s}$$

diminish with R

good enough for order-of-magnitude estimate

- Sum over cuts on internal lines of



expansion in $m_{i,j}/m_W$

$$\frac{1}{2\pi}\sum_k{}' \int_{t_k}^{\Lambda^2} ds' \frac{\Gamma^k_{ij}(s')}{s-s'} \approx -\frac{G_F^2 m_W^4}{16\pi^2}\left[1+\frac{m_i m_j}{m_W^2}\ln\frac{\Lambda^2}{m_W^2} - \frac{m_i^2 m_j^2}{m_W^4}\left(\ln\frac{m_W^2}{m_i m_j} - \frac{1}{4}\right)\right]\frac{2}{s}$$

$\sum'_k \equiv \sum_{k=1}^{2} - \sum_{k=3}^{4}$

$t_k = (m_i+m_j)^2,\ 4m_W^2,\ (m_i+m_W)^2,\ (m_j+m_W)^2$

Solution

If $m_\nu \sim O(1)\ \mathrm{eV}$, no need to tell which generation it refers to

$$\Delta m_{21}^2 \equiv m_2^2 - m_1^2 = (7.55^{+0.20}_{-0.16}) \times 10^{-5}\ \ \mathrm{eV}^2$$

$$\Delta m_{32}^2 \equiv m_3^2 - m_2^2 = (2.424 \pm 0.03) \times 10^{-3}\ \ \mathrm{eV}^2$$

- Insert $G_F = \sqrt{2}g^2/(8m_W^2)$ and $m_W = gv/2$.

set to m_ν

$$\frac{1}{2\pi}{\sum_k}' \int_{t_k}^{\Lambda^2} ds' \frac{\Gamma_{ij}(s')}{s-s'} \approx -\frac{1}{16\pi^2}\left[4\left(\frac{g}{2\sqrt{2}}\right)^4 + 2\left(\frac{g}{2\sqrt{2}}\right)^2 \frac{m_i m_j}{v^2}\ln\frac{\Lambda^2}{m_W^2} - \frac{m_i^2 m_j^2}{v^4}\left(\ln\frac{m_W^2}{m_i m_j} - \frac{1}{4}\right)\right]\frac{1}{s}$$

$$M_{ij}(s) \approx -\frac{1}{16\pi^2}\left[4\left(\frac{g}{2\sqrt{2}}\right)^4 + \frac{1}{4\pi^2}\left(\frac{g}{2\sqrt{2}}\right)^2 \frac{m_{\mathcal{L}}^2 m_\ell^2}{v^4} - \frac{3}{64}\left(\frac{1}{16\pi^2}\right)^2 \frac{m_{\mathcal{L}}^4 m_\ell^4}{v^8} + O\left(g^4 \frac{m_i^2 m_j^2}{v^4}\right)\right]\frac{1}{s}$$

- Establish solution (g can vary arbitrarily in mathematical viewpoint)
- To probe how small neutrino mass is, consider μe mixing
- $O(g^4)$ terms exactly identical

Neutrino mass and new physics scale

- Equality of $O(g^0)$ terms

no need of new physics scale

$$m_\nu^2\sqrt{\ln\frac{m_W^2}{m_\nu^2}} \approx \frac{\sqrt{3}}{128\pi^2}\frac{m_\mu^2 m_e^2}{v^2} \quad \Rightarrow \quad m_\nu \approx 3\ \mathrm{eV}$$

$m_\nu < 0.9\ \mathrm{eV\ at\ 90\%\ CL}$

by Katrin Collaboration

measure of 3-loop integral

like Majorana mass

- Equality of $O(g^2)$ terms

$$m_\nu^2 \ln\frac{\Lambda}{m_W} \approx \frac{1}{16\pi^2}\frac{m_\mu^2 m_e^2}{v^2} \quad \Rightarrow \quad \ln\frac{\Lambda}{m_W} \sim \overset{\downarrow}{O(1)}\sqrt{\ln\frac{m_W^2}{m_\nu^2}}$$

- Large new physics (restoration) scale Λ linked to small neutrino mass
- No definite prediction for Λ; need to compute all diagrams
- Crude guesstimate $\Lambda \gtrsim O(100)\ \mathrm{TeV}$

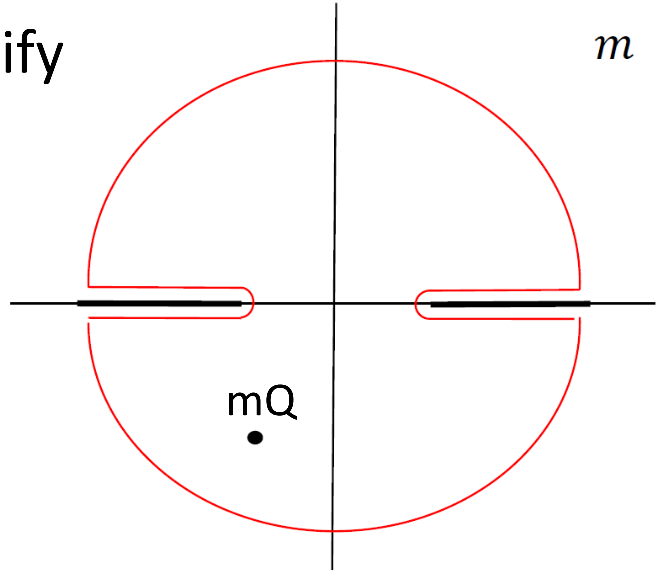
Top mass

Heaviest particle in SM

Assume massless 1st generation quarks; derive masses of heavier quarks one by one using heavy quark decay widths; get ms ~ 0.1 GeV, mc ~ 1.4 GeV, mb ~ 4 GeV

Framework

heavy quark mQ to Justify
perturbative evaluation



- Consider box diagram for tu mixing at mQ
- Strong interaction involved

big circle
contributions
cancel, because

$$\mathrm{Im}\Pi_{ij}(m) \rightarrow \mathrm{Im}\Pi_{ij}^{\mathrm{box}}(m)$$

unknowns to
be solved

perturbative
inputs from
box diagrams

$$\int_{M_{ij}^2}^{R^2} \frac{\mathrm{Im}\Pi_{ij}(m)}{m_Q^2 - m^2} dm^2 = \int_{m_{ij}^2}^{R^2} \frac{\mathrm{Im}\Pi_{ij}^{\mathrm{box}}(m)}{m_Q^2 - m^2} dm^2$$

branch cuts along
both m > 0, m < 0

due to analyticity

3 channels $ij = db,\ sb$ and bb

hadronic thresholds

$$M_{db} = m_\pi + m_B,\ M_{sb} = m_K + m_B \text{ and } M_{bb} = 2m_B$$

quark-level thresholds

$$m_{db} = m_d + m_b,$$
$$m_{sb} = m_s + m_b \text{ and } m_{bb} = 2m_b.$$

Box diagram inputs

- Box diagrams generate (V-A)(V-A), (S-P)(S-P) structures
- Focus on the former

intermediate quark masses

LO QCD

$$
\begin{aligned}
\Gamma_{ij}^{\text{box}}(m_Q) \;\propto\; & \frac{C_2(m_Q)}{m_Q^4}\,\frac{\sqrt{[m_Q^2-(m_i+m_j)^2][m_Q^2-(m_i-m_j)^2]}}{(m_W^2-m_i^2)(m_W^2-m_j^2)} \\
& \times\left\{2\left(m_W^4+\frac{m_i^2m_j^2}{4}\right)[m_Q^2-(m_i+m_j)^2][m_Q^2-(m_i-m_j)^2]\right. \\
& \left. -3m_W^2m_Q^2(m_i^2+m_j^2)(m_Q^2-m_i^2-m_j^2)\right\},
\end{aligned}
$$

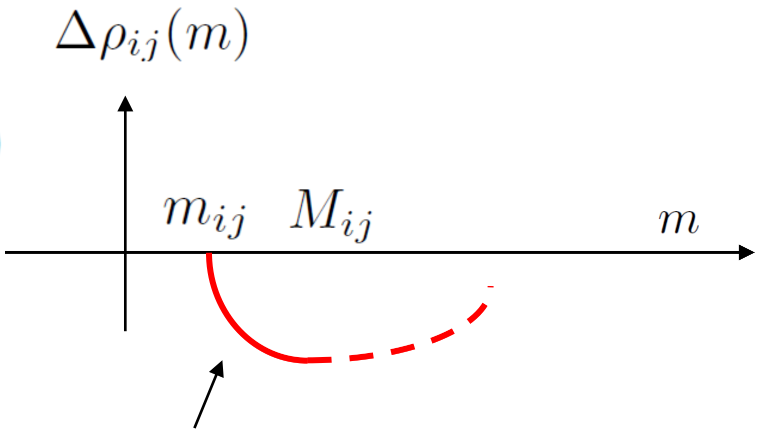
W boson mass

Initial conditions

- Move RHS to LHS, $\Delta\rho_{ij}(m) \equiv \mathrm{Im}\Pi_{ij}(m) - \mathrm{Im}\Pi_{ij}^{\mathrm{box}}(m)$

extended to infinity

$$\int_{m_{ij}^2}^{\infty} \frac{\Delta\rho_{ij}(m)}{m_Q^2 - m^2} dm^2 = 0$$



initial condition

$-\mathrm{Im}\Pi_{ij}^{\mathrm{box}}(m)$

- Threshold behaviors around $m_Q \sim m_{ij}$

$$\Gamma_{db}^{\mathrm{box}}(m_Q) \sim \frac{(m_Q^2 - m_b^2)^3}{m_Q^4}, \qquad m_d = 0$$

$$\Gamma_{sb}^{\mathrm{box}}(m_Q) \sim \frac{\sqrt{[m_Q^2 - (m_b + m_s)^2][m_Q^2 - (m_b - m_s)^2]}^3}{m_Q^4}$$

$$\Gamma_{bb}^{\mathrm{box}}(m_Q) \sim \frac{\sqrt{m_Q^2 - 4m_b^2}^3}{m_Q}.$$

odd power in m

governed by 1st term
in curly brackets
2nd term down by $(m_i^2 + m_j^2)/m_W^2$

Integrands

- Motivated by threshold behaviors, choose integrands (to simplify initial conditions)

suppress low-m residues like D meson mass or $m = \pm(m_i + m_j)$ relative to $m = \pm m_Q$

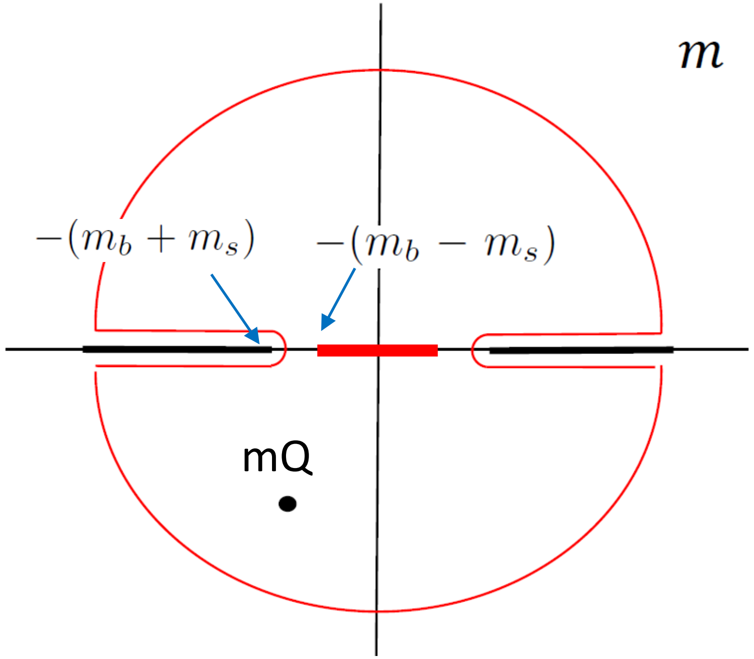
$$\mathrm{Im}\Pi_{db}(m) = \frac{m^4\Gamma_{db}(m)}{(m^2 - m_b^2)^2},$$

alleviate divergent behaviors in numerators

$$\mathrm{Im}\Pi_{sb}(m) = \frac{m^4\Gamma_{sb}(m)}{[m^2-(m_b+m_s)^2]^2\sqrt{m^2-(m_b-m_s)^2}^3}$$

$$\mathrm{Im}\Pi_{bb}(m) = \frac{m\Gamma_{bb}(m)}{m^2 - 4m_b^2},$$

additional branch cut does not contribute



odd power of m due to odd function $\Gamma_{bb}^{\mathrm{box}}(m)$ in m

- Definitions of $\mathrm{Im}\Pi_{ij}^{\mathrm{box}}(m)$ are self-evident

Polynomial expansion

arbitrary scale

- Introduce dimensionless variables, $m_Q^2 - 4m_b^2 = \underset{}{\overset{\downarrow}{u}}\Lambda$, $m^2 - 4m_b^2 = v\Lambda$

$$\int_0^\infty dv \frac{\Delta\rho(v)}{u-v} = 0$$

$\Delta\rho(v) \to 0$ at large v, because $\mathrm{Im}\Pi(m) \to \mathrm{Im}\Pi^{\mathrm{p}}(m)$

power series in $1/u$ using $1/(u-v) = \sum_{i=1}^{\infty} v^{i-1}/u^i$

- Start with case of N vanishing coefficients, N large

contained in $L_0^{(\alpha)}(v), L_1^{(\alpha)}(v), \cdots, L_{N-1}^{(\alpha)}(v)$

$$\int_0^\infty dv v^{i-1} \Delta\rho(v) = 0, \quad i = 1, 2, 3 \cdots, N$$

- Imply expansion in generalized Laguerre polynomials because of orthogonality

weight

$$\Delta\rho(v) = \sum_{j=N}^{N'} a_j \underline{v^\alpha e^{-v}} L_j^{(\alpha)}(v), \quad \underset{\uparrow}{N'} > N \qquad \int_0^\infty \underline{y^\alpha e^{-y}} L_m^{(\alpha)}(y) L_n^{(\alpha)}(y) dy = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{mn}$$

fixed by initial condition in principle, needs not be infinite

Large N limit

- Large j approximation, subject to correction of $1/\sqrt{j}$

$$L_j^{(\alpha)}(v) \approx j^{\alpha/2} v^{-\alpha/2} e^{v/2} J_\alpha(2\sqrt{jv})$$

- Solution

arbitrary degree and scale appear in ratio

$$\Delta\rho(m) \approx \sum_{j=N}^{N'} a_j \sqrt{\frac{j(m^2-4m_b^2)}{\Lambda}}^{\alpha} e^{-(m^2-4m_b^2)/(2\Lambda)} J_\alpha\left(2\sqrt{\frac{j(m^2-4m_b^2)}{\Lambda}}\right)$$

- Scaling variable $\omega \equiv \sqrt{N/\Lambda}$, large N limit $N'/\Lambda = \omega^2 + (N'-N)/N \approx \omega^2$

$$J_\alpha(2\sqrt{j(m^2-4m_b^2)/\Lambda}) \approx J_\alpha(2\omega\sqrt{m^2-4m_b^2}) \qquad e^{-(m^2-4m_b^2)/(2\Lambda)} = e^{-\omega^2(m^2-4m_b^2)/(2N)}$$

≈ 1

$$\Delta\rho(m) \approx \underset{\uparrow}{y} \left(\omega\sqrt{m^2-4m_b^2}\right)^{\alpha} J_\alpha\left(2\omega\sqrt{m^2-4m_b^2}\right)$$

$$\sum_{j=N}^{N'} a_j$$

solution in terms of single Bessel function

Solutions

- General form

originating from large circle radius R

$$\Delta\rho_{ij}(m_Q) \approx y_{ij}\left(\omega\sqrt{m_Q^2-(m_i+m_j)^2}\right)^{\alpha_{ij}} J_{\alpha_{ij}}\left(2\omega\sqrt{m_Q^2-(m_i+m_j)^2}\right)$$

arbitrary scale from scaling integration variable m^2

- Insensitivity to ω achieved by

vanishing to get discrete roots of m_Q

Taylor expansion

$$\Delta\rho_{ij}(m_Q) = \Delta\rho_{ij}(m_Q)|_{\omega=\bar{\omega}_{ij}} + \frac{d\Delta\rho_{ij}(m_Q)}{d\omega}\Big|_{\omega=\bar{\omega}_{ij}}(\omega-\bar{\omega}_{ij}) + \frac{1}{2}\frac{d^2\Delta\rho_{ij}(m_Q)}{d\omega^2}\Big|_{\omega=\bar{\omega}_{ij}}(\omega-\bar{\omega}_{ij})^2 + \cdots$$

fitted to initial conditions
to fix $\bar{\omega}_{ij}$, α_{ij}, y_{ij}

minimal to maximize stability window in ω

Parameter fixing

$$m_d = 0 \qquad m_s = 0.1 \text{ GeV} \qquad m_b = 4.15 \text{ GeV}$$

$$m_\pi = 0.14 \text{ GeV} \quad m_K = 0.49 \text{ GeV}, \ m_B = 5.28 \text{ GeV}$$

- Initial conditions around $m_Q \sim m_{ij}$

$$\begin{aligned}\Delta\rho_{db}(m_Q) &\sim m_Q^2 - m_b^2, \\ \Delta\rho_{sb}(m_Q) &\sim [m_Q^2 - (m_b + m_s)^2]^{-1/2} \\ \Delta\rho_{bb}(m_Q) &\sim (m_Q^2 - 4m_b^2)^{1/2}.\end{aligned}$$

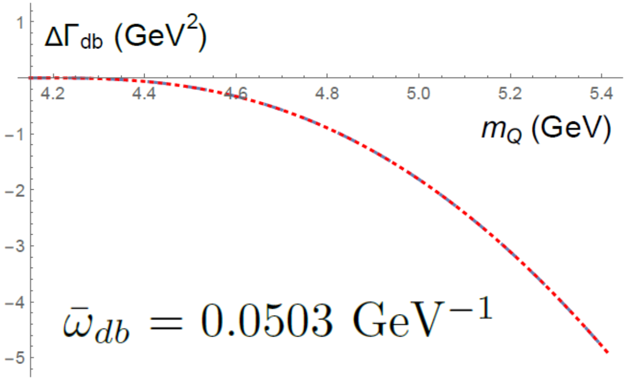
clear why considering complicated integrands: to have simple power of $m_Q^2 - (m_i + m_j)^2$

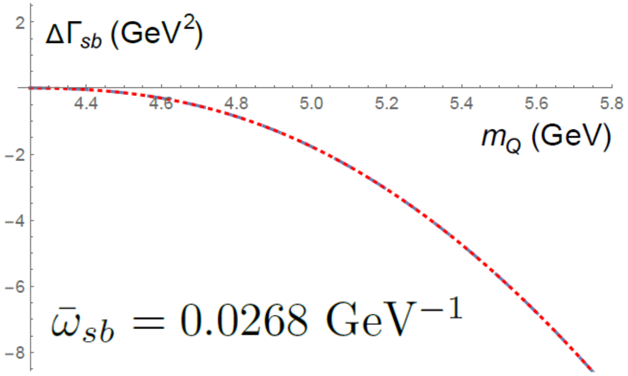
$$\Rightarrow \quad \alpha_{db} = 1, \quad \alpha_{sb} = -1/2, \quad \alpha_{bb} = 1/2$$

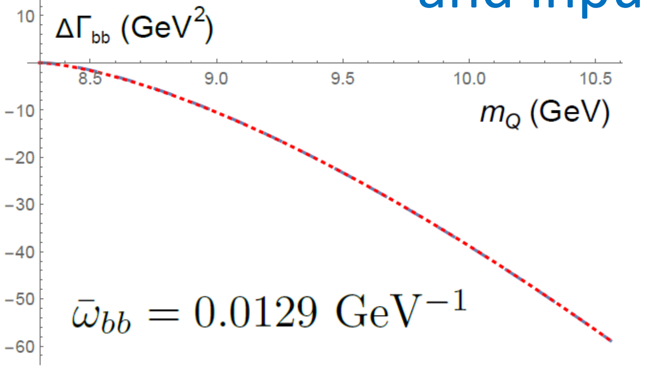
- Boundary conditions $\Delta\rho_{ij}(m_Q)$ set coefficients

$$y_{ij} = -\mathrm{Im}\Pi_{ij}^{\mathrm{box}}(M_{ij}) \left[\left(\omega\sqrt{M_{ij}^2 - (m_i + m_j)^2}\right)^{\alpha_{ij}} J_{\alpha_{ij}} \left(2\omega\sqrt{M_{ij}^2 - (m_i + m_j)^2}\right) \right]^{-1}$$

comparison of fitted results and inputs





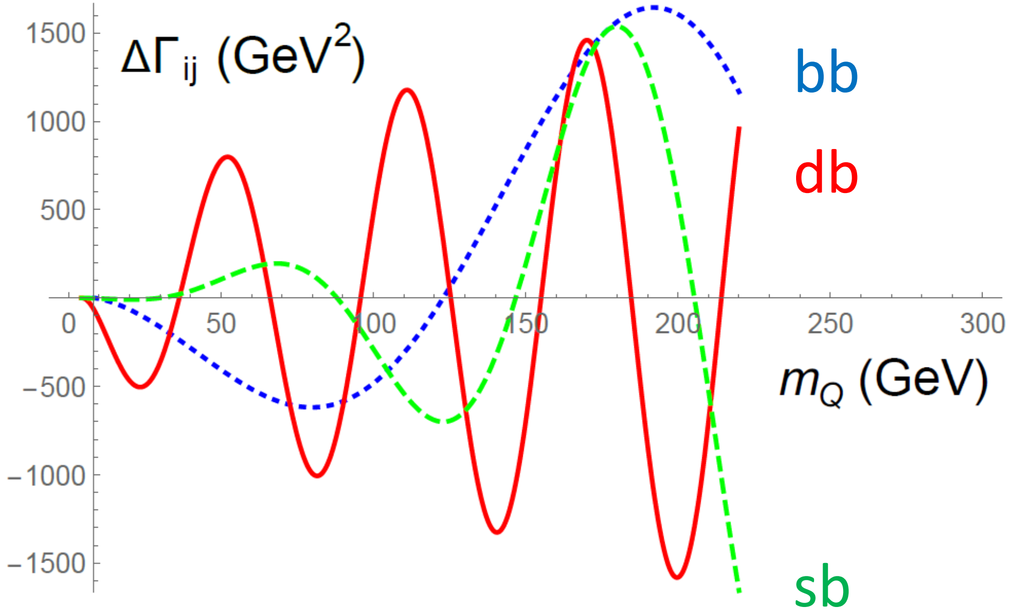


Roots

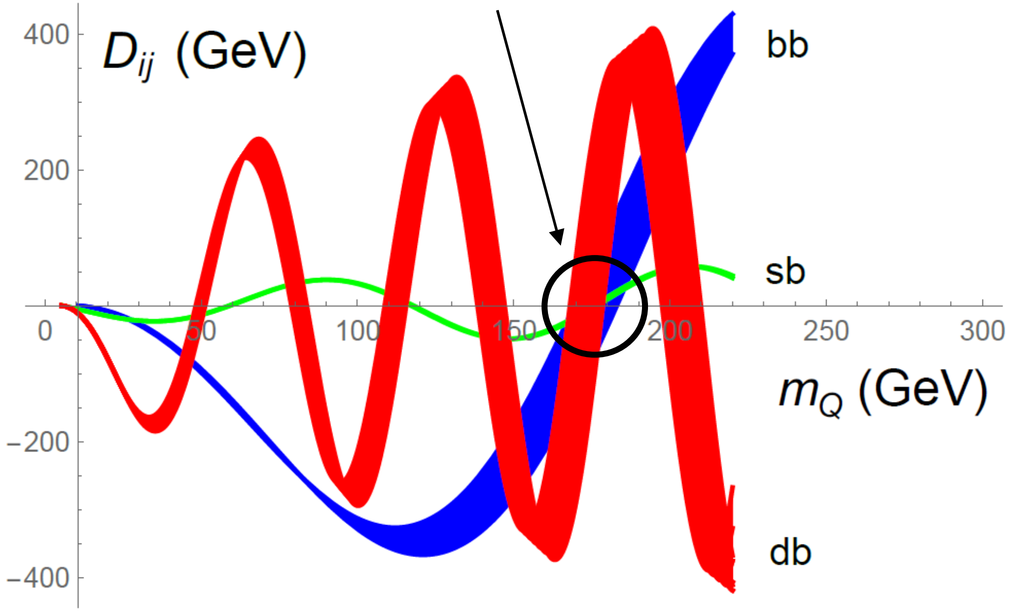
- Solutions of unknowns

higher roots, larger 2nd derivative

3 derivatives first vanish simultaneously at $m_t = (173 \pm 3)$ GeV



1st peak of bb, 2nd peak of sb, 3rd peak of db overlap around mQ ~ 180 GeV!



$$D_{ij}(m_Q) \equiv \frac{d}{d\omega} \frac{J_{\alpha_{ij}}\left(2\omega\sqrt{m_Q^2 - (m_i + m_j)^2}\right)}{J_{\alpha_{ij}}\left(2\omega\sqrt{M_{ij}^2 - (m_i + m_j)^2}\right)}\Bigg|_{\omega=\bar{\omega}_{ij}}$$

uncertainties from $m_b = (4.16 \pm 0.01)$ GeV and different ways of fixing $\bar{\omega}_{ij}$

Summary

- Mass hierarchy and mixing patterns explained by dispersive constraints
- Possible that SM contains only three fundamental parameters (gauge couplings)
- Other parameters, governing interplay among generations of fermions, are determined by SM dynamics itself
- Then SM flavor structure can be understood in dynamical way
- If our explanation is correct, it sheds light on model building for new physics