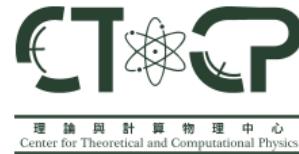
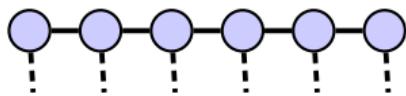


# Direct calculation of parton distribution functions with time evolved tensor network states

**Manuel Schneider**

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[arXiv:2504.07508]

[arXiv:2409.16996]

SQAI-NCTS Workshop on Quantum Technologies and Machine Learning  
Taipei  
26 August 2025

## Collaborators



Mari Carmen Bañuls



Krzysztof Cichy



C.-J. David Lin

## Outline

1 Parton Distribution Functions

2 Tensor Network States

3 Schwinger Model

4 Results

5 Summary & Outlook

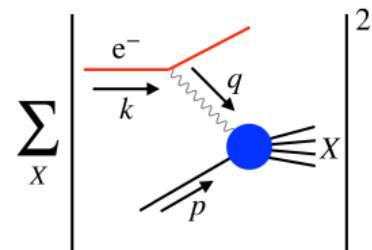
## Parton Distribution Functions

- ▶ PDF: probability of constituent with momentum fraction  $\xi$



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- ▶ large momentum transfer  $Q^2 = -q^2$
- ▶ Bjorken parameter  $\xi = \frac{Q^2}{2P \cdot q}$

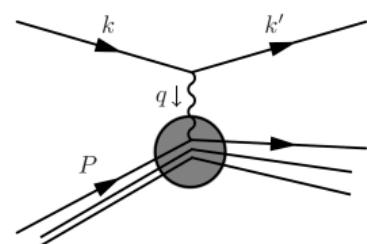
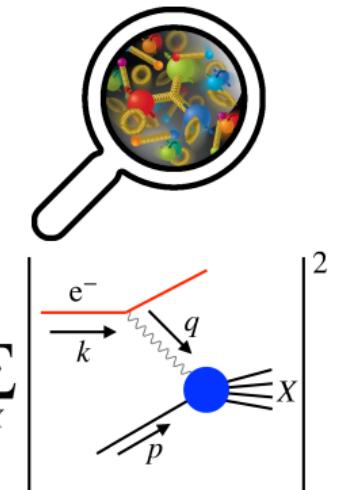


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$$\sigma(\xi, Q^2) = \hat{\sigma}(\xi, Q^2) f_\psi(\xi)$$

experiment →      perturbative ↑      non-perturbative ←



[Schwartz 2014]

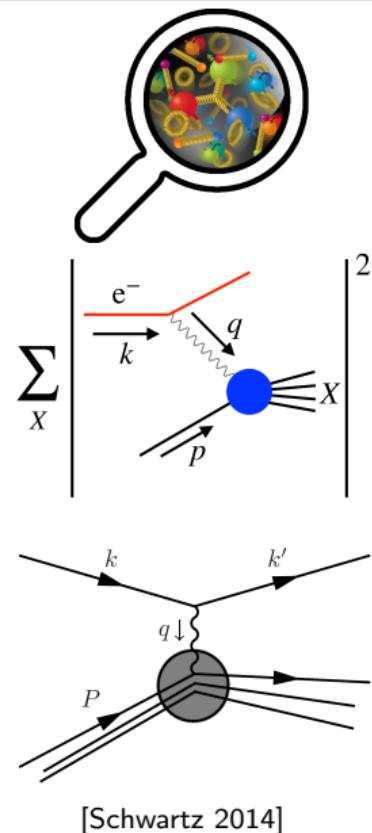
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## Parton Distribution Functions

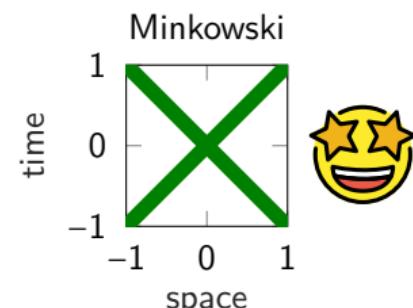
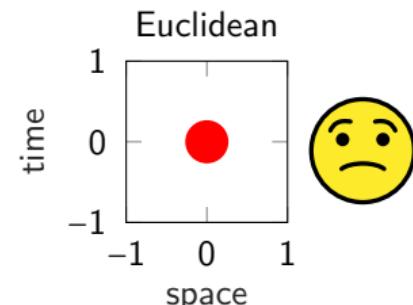
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- ▶  $z^-$ : lightcone coordinate
- ▶ lattice QCD in Euclidean space: lightcone → point
- ▶ Hamiltonian formalism: lightcone in Minkowski space  
 → use tensor network states/quantum devices



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## Tensor Network States

- ▶ generic state scales exponentially

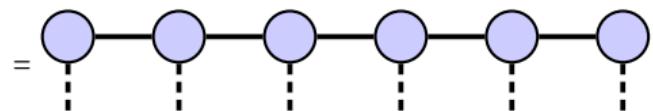
$$|\psi\rangle = \sum_{s_1, s_2, \dots, s_N} \Psi^{s_1 s_2 \dots s_N} |s_1\rangle \otimes |s_2\rangle \otimes \dots \otimes |s_N\rangle$$

## Tensor Network States

- ▶ generic state scales exponentially
- ▶ tensor network state as ansatz
- ▶ 1d: matrix product state (MPS)

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$$\Psi^{s_1 s_2 \dots s_N} = \sum_{\{i_x\}} A_{i_1}^{1, s_1} \cdot A_{i_1, i_2}^{2, s_2} \cdot A_{i_2, i_3}^{3, s_3} \dots A_{i_{N-1}}^{N, s_N}$$



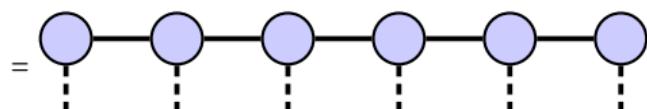
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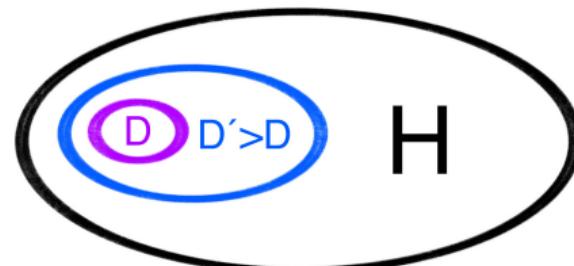
$$\Psi^{s_1 s_2 \dots s_N} \approx \sum_{\{i_x\}=1}^D A_{i_1}^{1, s_1} \cdot A_{i_1, i_2}^{2, s_2} \cdot A_{i_2, i_3}^{3, s_3} \dots A_{i_{N-1}}^{N, s_N}$$



- ▶ 1d: matrix product state (MPS)

- ▶ truncation to bond dimension D

- ▶ polynomial resource scaling



## Tensor Network States

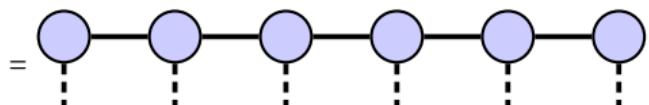
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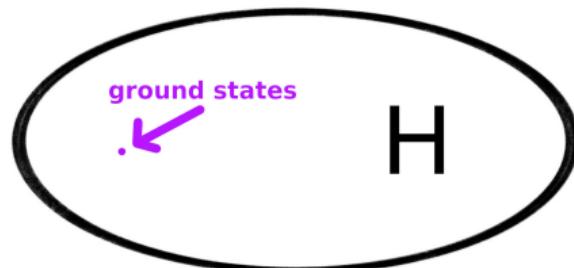
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- truncation to bond dimension D

- polynomial resource scaling

- good for ground states and low excited states [Hastings 2007]



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## Schwinger Model [Hamer et al. 1997]

- ▶ quantum electrodynamics in 1+1 dimensions,  $U(1)$  symmetry
- ▶ fermion couples to gauge boson → partons
- ▶ bound states → hadrons [Bañuls et al. 2013]
- ▶ scattering → PDF [Dai et al. 1995]
- ▶ Lagrange density:

$$\mathcal{L} = \bar{\Psi}(i\cancel{\partial} - g\cancel{A} - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_0\rho$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

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- ▶ for TN/QC: transform action into spin-model Hamiltonian:

$$H = x \sum_{n=0}^{N-2} [\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+] + \frac{\mu}{2} \sum_{n=0}^{N-1} [1 + (-1)^n \sigma_n^z] + \sum_{n=0}^{N-2} \left[ \sum_{k=0}^n Q_k \right]^2$$

$$\left( x = \frac{1}{a^2 g^2}, \mu = \frac{2m}{ag^2}, Q_k = \frac{1}{2} ((-1)^k + \sigma_k^z) + q_k \right)$$

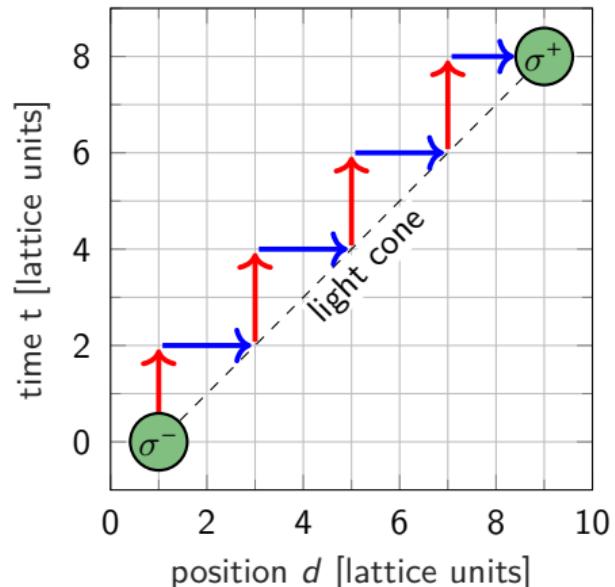
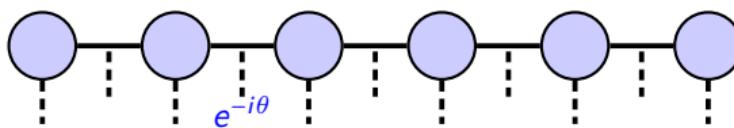
## Lightfront matrix elements

$$\langle P | \bar{\Psi}(z^-) \gamma^+ W(z^- \leftarrow 0) \Psi(0) | P \rangle \\ \rightarrow \langle P | \sigma^+(z^-) W_{z^- \leftarrow 0} \sigma^-(0) | P \rangle + \dots$$

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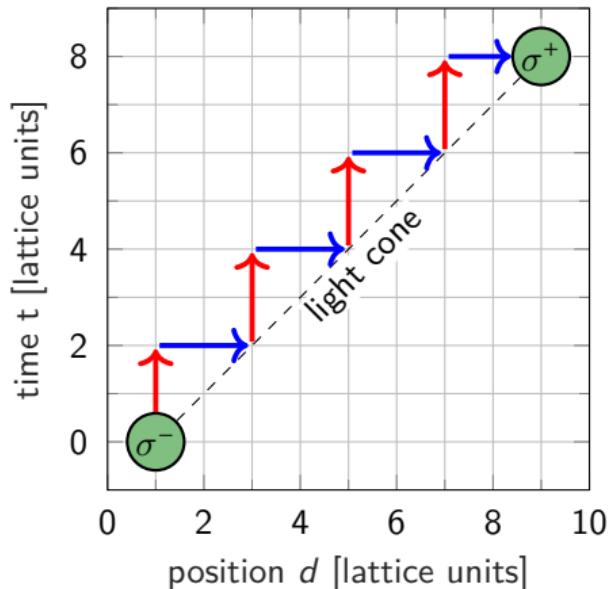
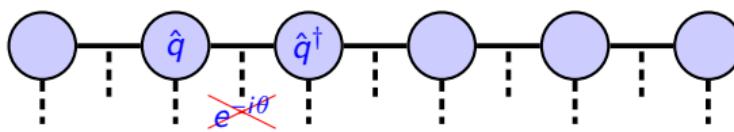
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→ small **time-** and **space-like** steps
- ▶ time evolution:  
 $e^{-i\tau H} \approx (e^{-i\delta\tau H_{eo}} e^{-i\delta\tau H_{oe}} e^{-i\delta\tau H_L})^{N_\tau}$
- ▶ spatial evolution:  
**change electric field** along the path  
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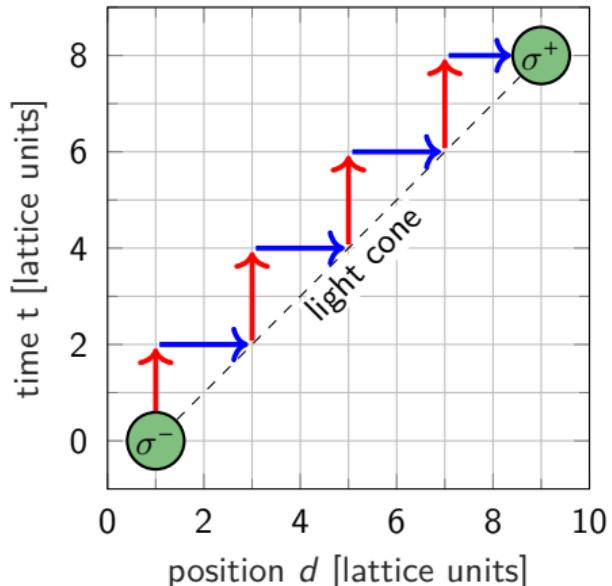
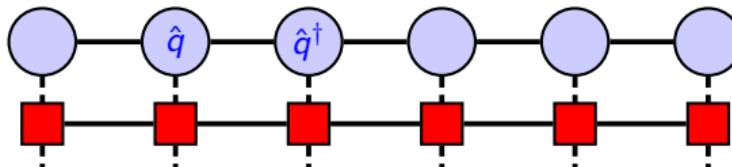
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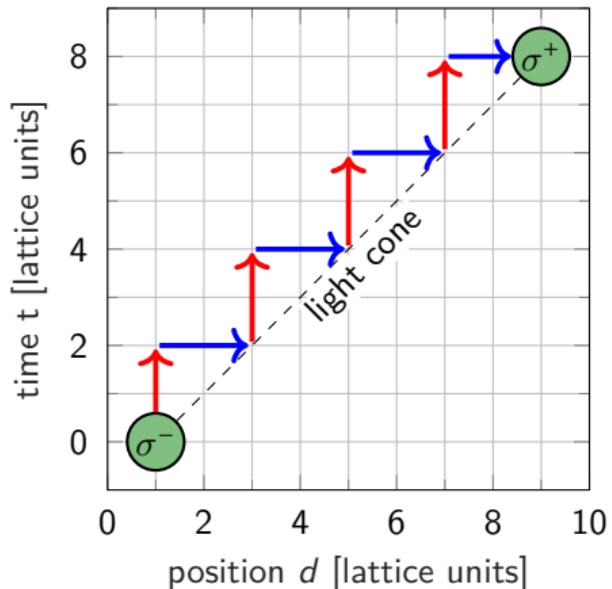
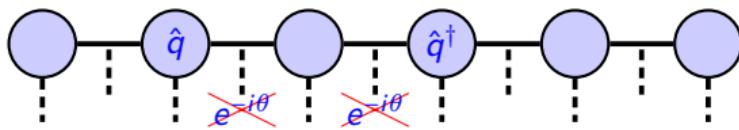
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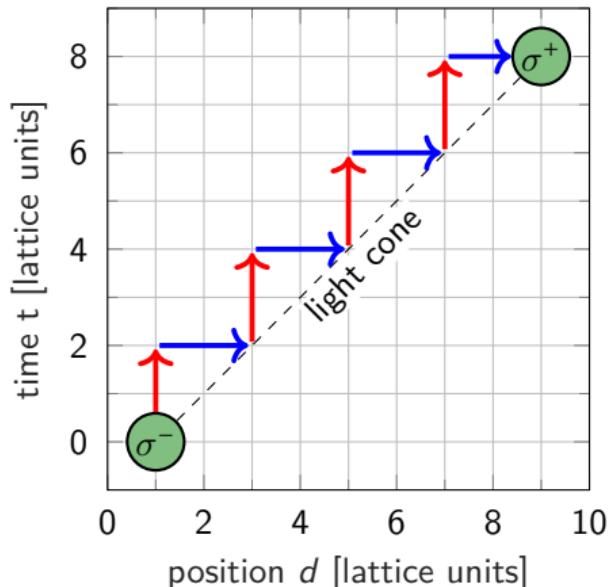
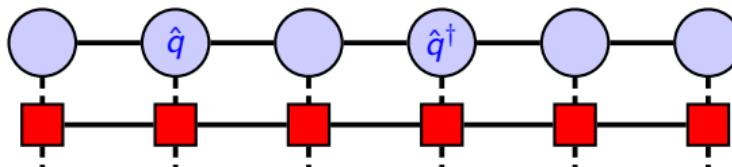
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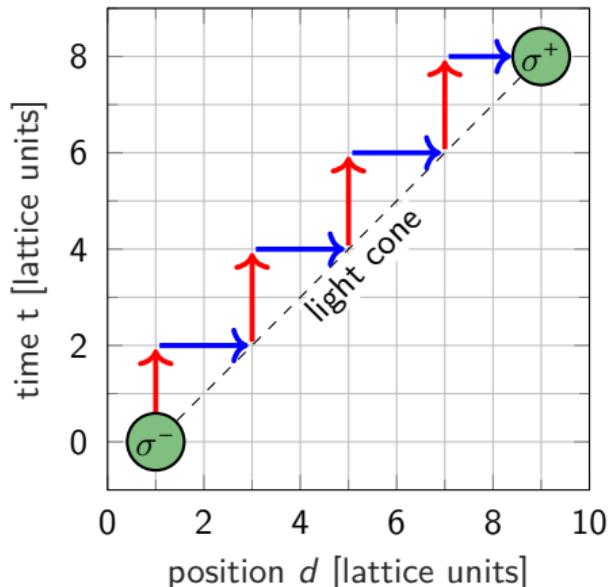
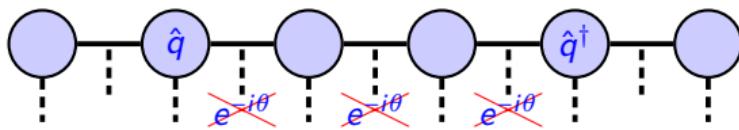
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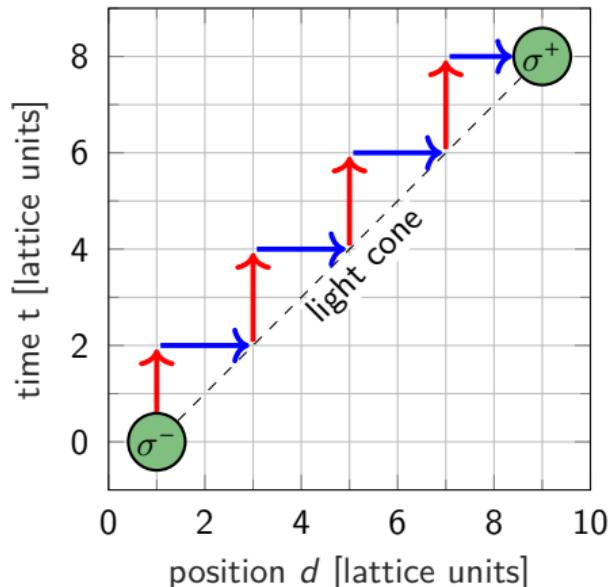
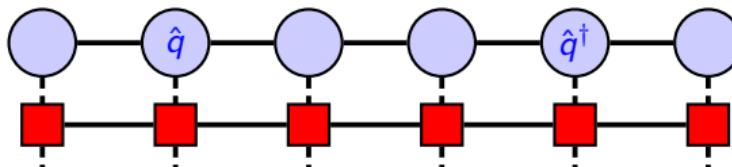
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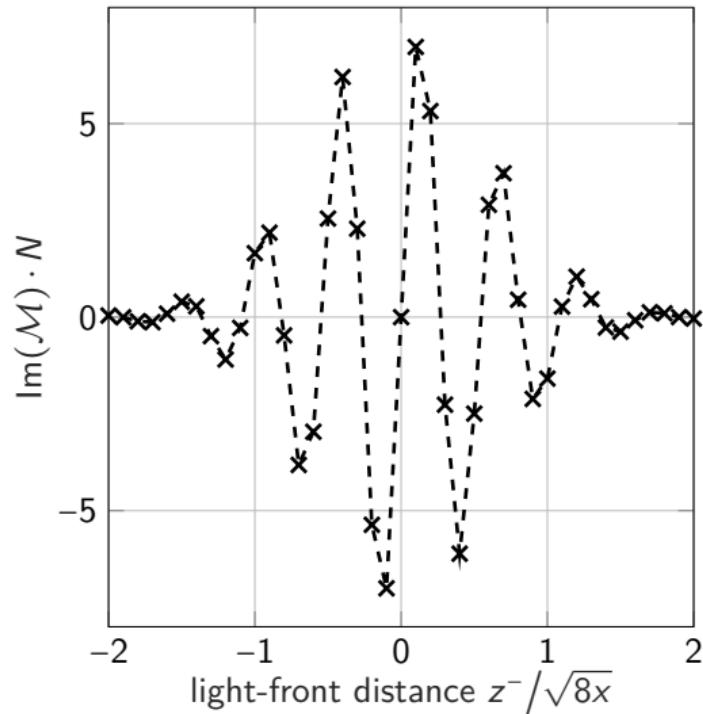
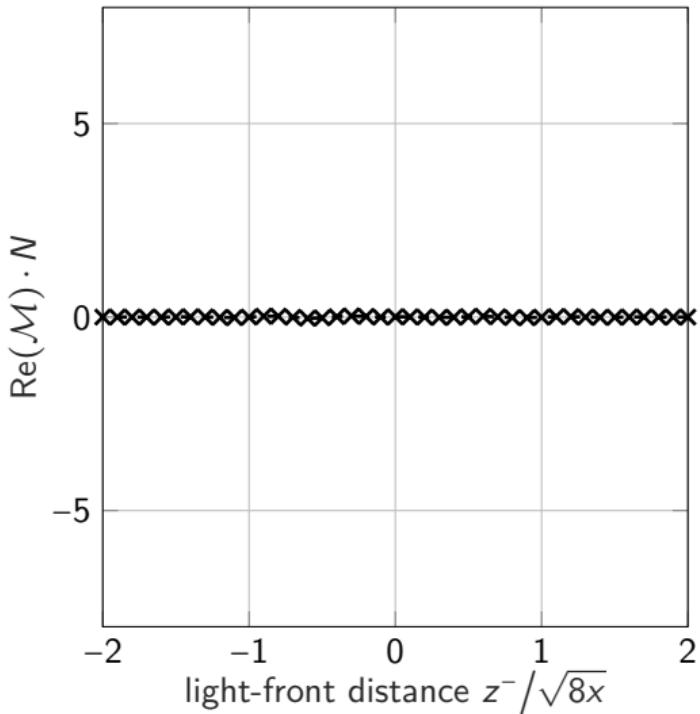
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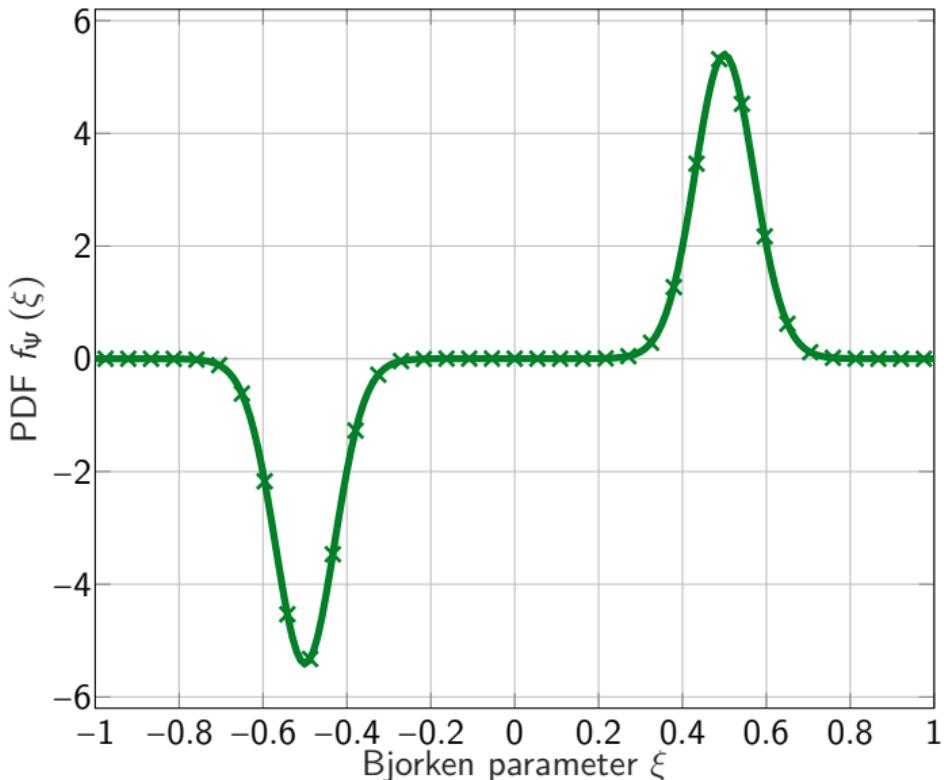
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## matrix elements

$\tilde{m} = 10; x = 100; D = 80; N_T = 100; \tilde{V} = 100$



## PDF

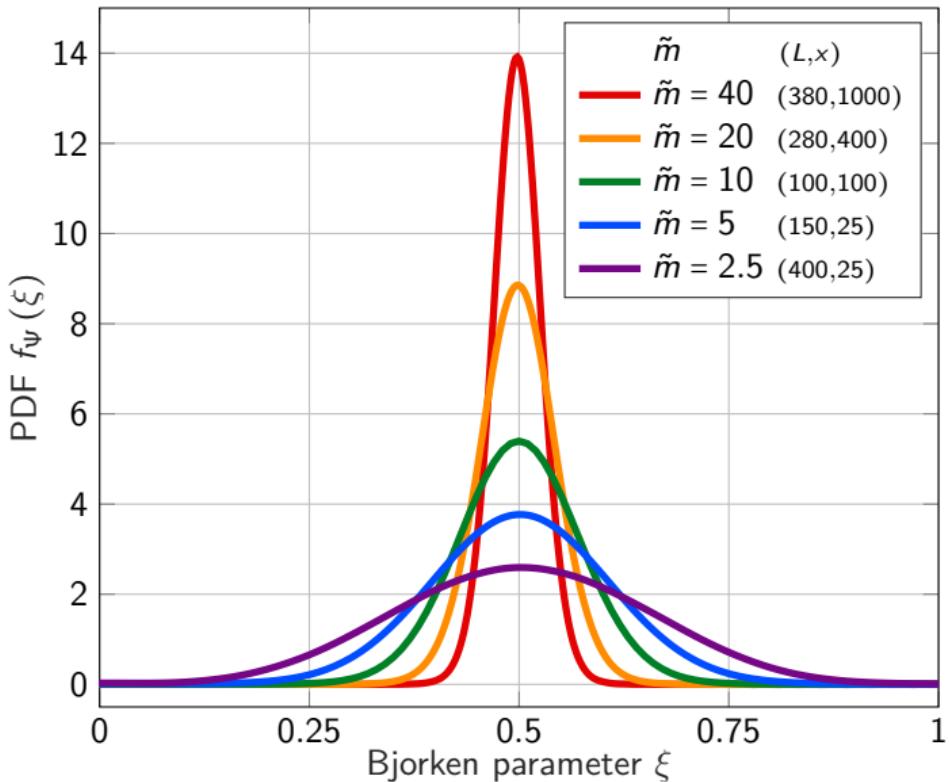
 $\tilde{m} = 10; x = 100; D = 80; N_\tau = 100; \tilde{V} = 100$ 

- $\xi > 0$ :  $f_\psi \approx$  symmetric around  $\xi = 0.5$
- antifermion PDF:

$$f_{\bar{\psi}}(\xi) = -f_\psi(-\xi)$$

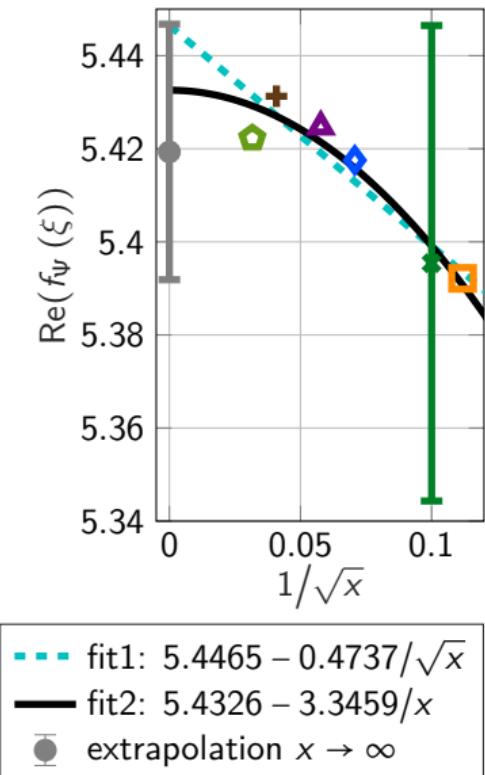
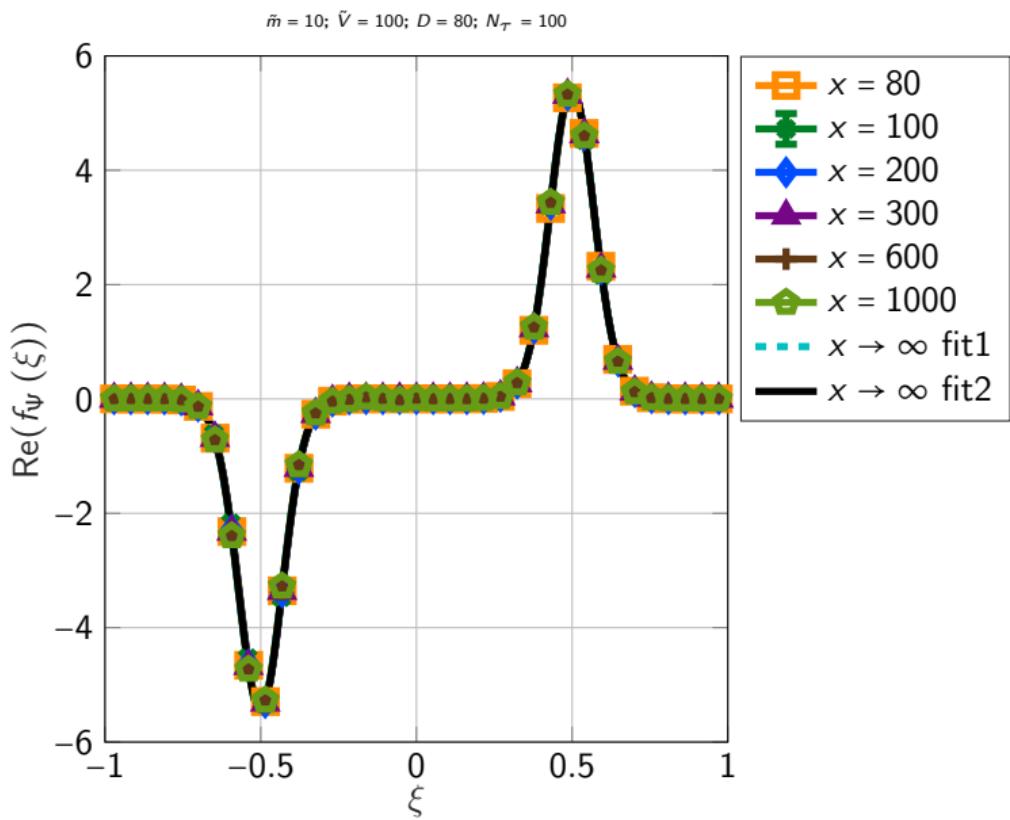
- $f_{\bar{\psi}}(\xi) = f_\psi(\xi)$   
⇒ meson ✓

## PDF

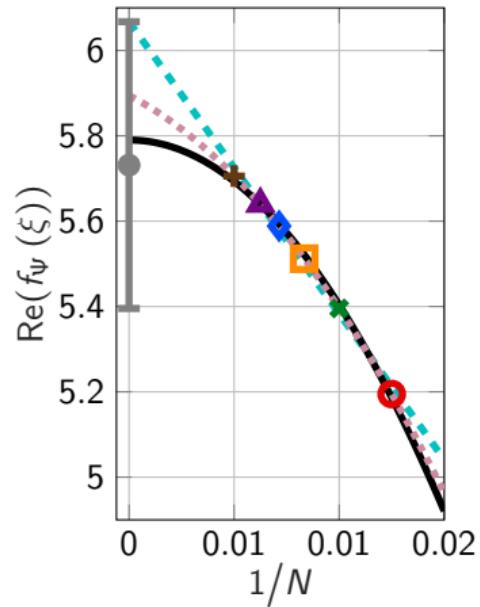
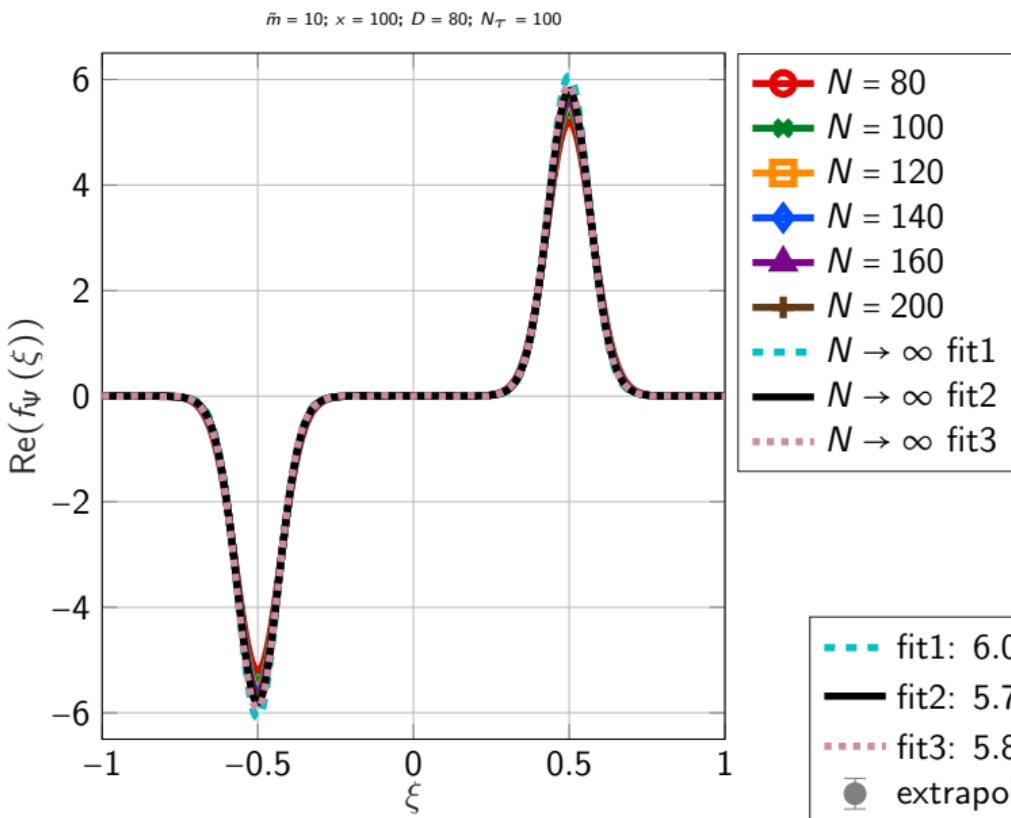
 $D = 80; N_T = 100$ 

- $\xi > 0: f_\psi \approx \text{symmetric around } \xi = 0.5$
- antifermion PDF:  
$$f_{\bar{\psi}}(\xi) = -f_\psi(-\xi)$$
- $f_{\bar{\psi}}(\xi) = f_\psi(\xi)$   
 $\Rightarrow \text{meson}$  ✓
- peak broadens with decreasing fermion mass ✓

## x-dependence



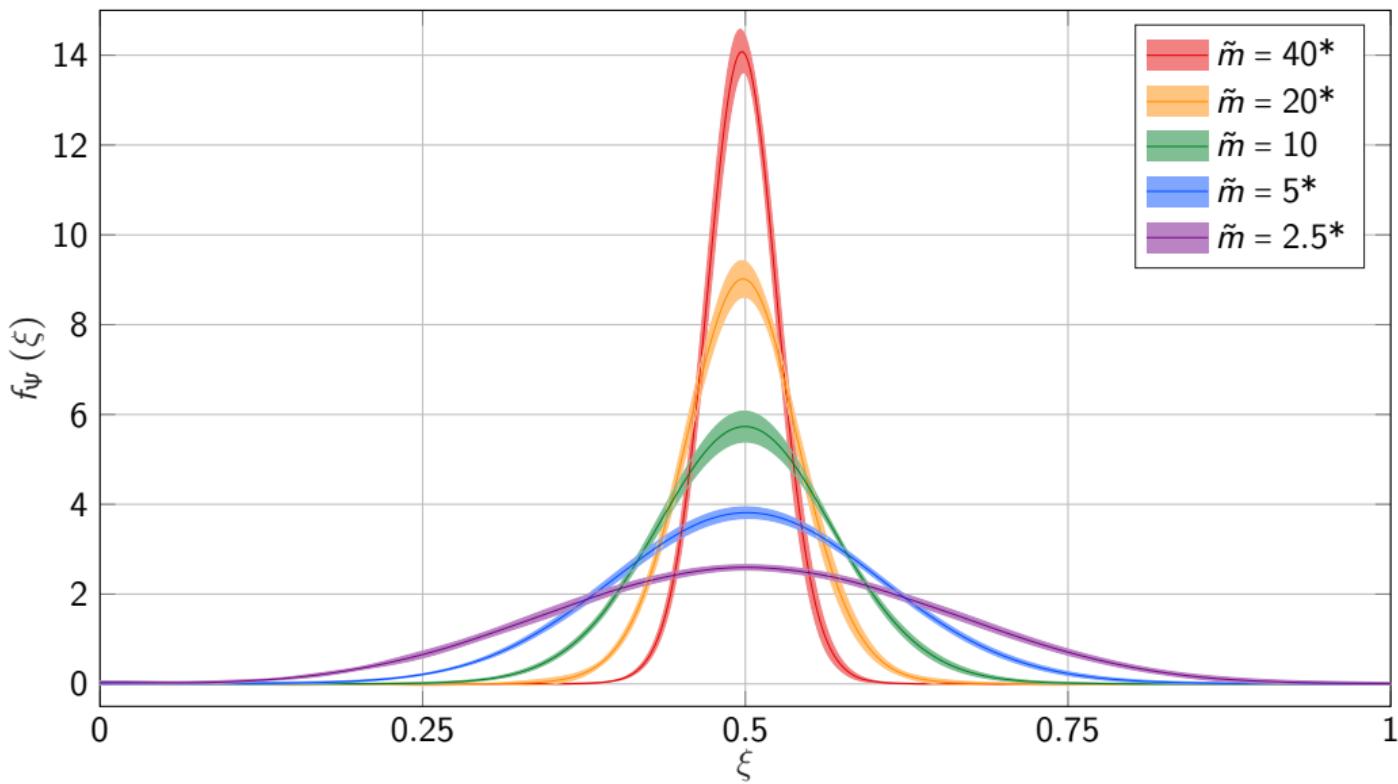
## $N$ -dependence



Legend:

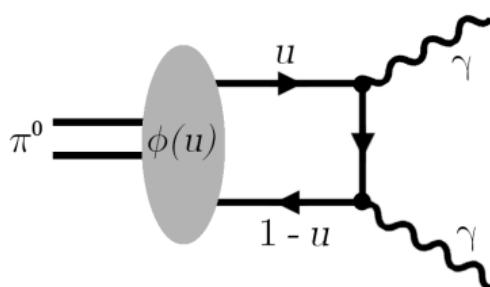
- $\textcolor{cyan}{- - -} \text{fit1: } 6.0674 - 68.4080/N$
- $\textcolor{black}{- - -} \text{fit2: } 5.7901 - 3865.5/N^2$
- $\textcolor{violet}{- - -} \text{fit3: } 5.8944 - 25.4653/N - 2441.7/N^2$
- $\textcolor{grey}{\bullet}$  extrapolation  $N \rightarrow \infty$

## PDF [\*preliminary]

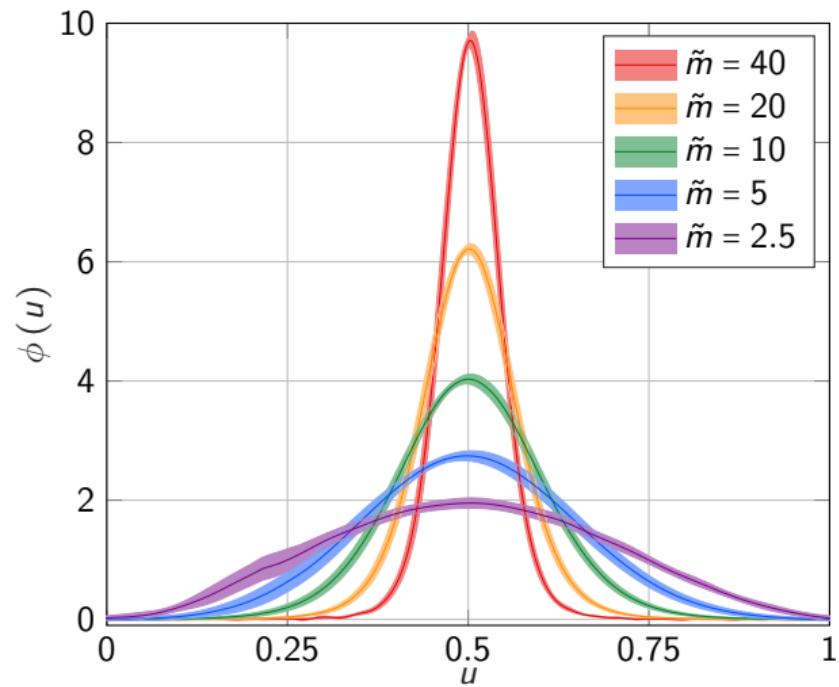


## lightcone distribution amplitude (LCDA) [preliminary]

LCDA: decay or hadronization



$$\pi^0 \rightarrow q(u)\bar{q}(1-u) \rightarrow \gamma\gamma$$



$$\text{if } \phi(u) = \int dz^- e^{iuP^+z^-} \langle 0 | \bar{\psi}(z^-) \gamma^+ W(z^- \leftarrow 0) \psi(0) | P \rangle$$

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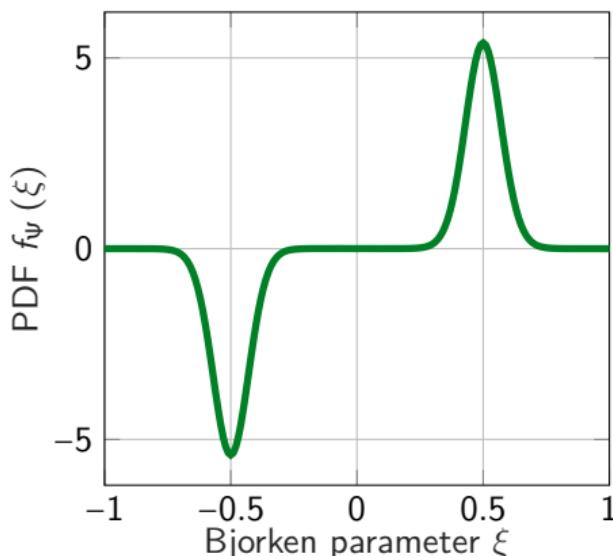
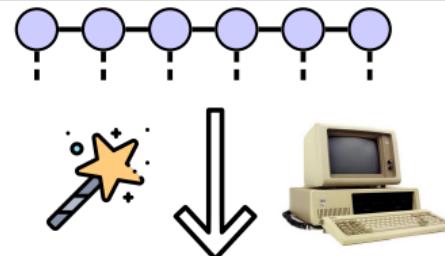
## Summary

Summary:

- ▶ PDF → universal structure of hadrons
- ▶ Euclidean space: lightcone → point 
- ▶ ⇒ use tensor network states / quantum devices
- ▶ Schwinger model:  
fermion- and anti-fermion-PDF for the vector meson



[arXiv:2504.07508]  
[arXiv:2409.16996]



Outlook:

- ▶ further lightcone observables
- ▶ same analysis for QCD 😊

<sup>1</sup>M. C. Bañuls, K. Cichy, C. J. D. Lin, and M. Schneider, “Parton distribution functions in the schwinger model from tensor network states,” arXiv e-prints, arXiv:2504.07508 (2025) doi:10.48550/arXiv.2504.07508.

<sup>2</sup>M. Schneider, M. C. Bañuls, K. Cichy, and C.-J. D. Lin, “Parton Distribution Functions in the Schwinger Model with Tensor Networks,” in Proceedings of the 41st international symposium on lattice field theory — pos(lattice2024), Vol. 466 (2025), p. 024, doi:10.22323/1.466.0024.

<sup>3</sup>M. D. Schwartz, *Quantum Field Theory and the Standard Model*, (Cambridge University Press, Mar. 2014), ISBN: 978-1-107-03473-0, 978-1-107-03473-0, doi:10.1017/9781139540940.

<sup>4</sup>M. B. Hastings, “An area law for one-dimensional quantum systems,” Journal of Statistical Mechanics: Theory & Exp. **2007**, 08024 (2007) doi:10.1088/1742-5468/2007/08/P08024.

<sup>5</sup>M. C. Bañuls, K. Cichy, J. I. Cirac, and K. Jansen, “The mass spectrum of the schwinger model with matrix product states,” JHEP **2013**, 158 (2013) doi:10.1007/JHEP11(2013)158.

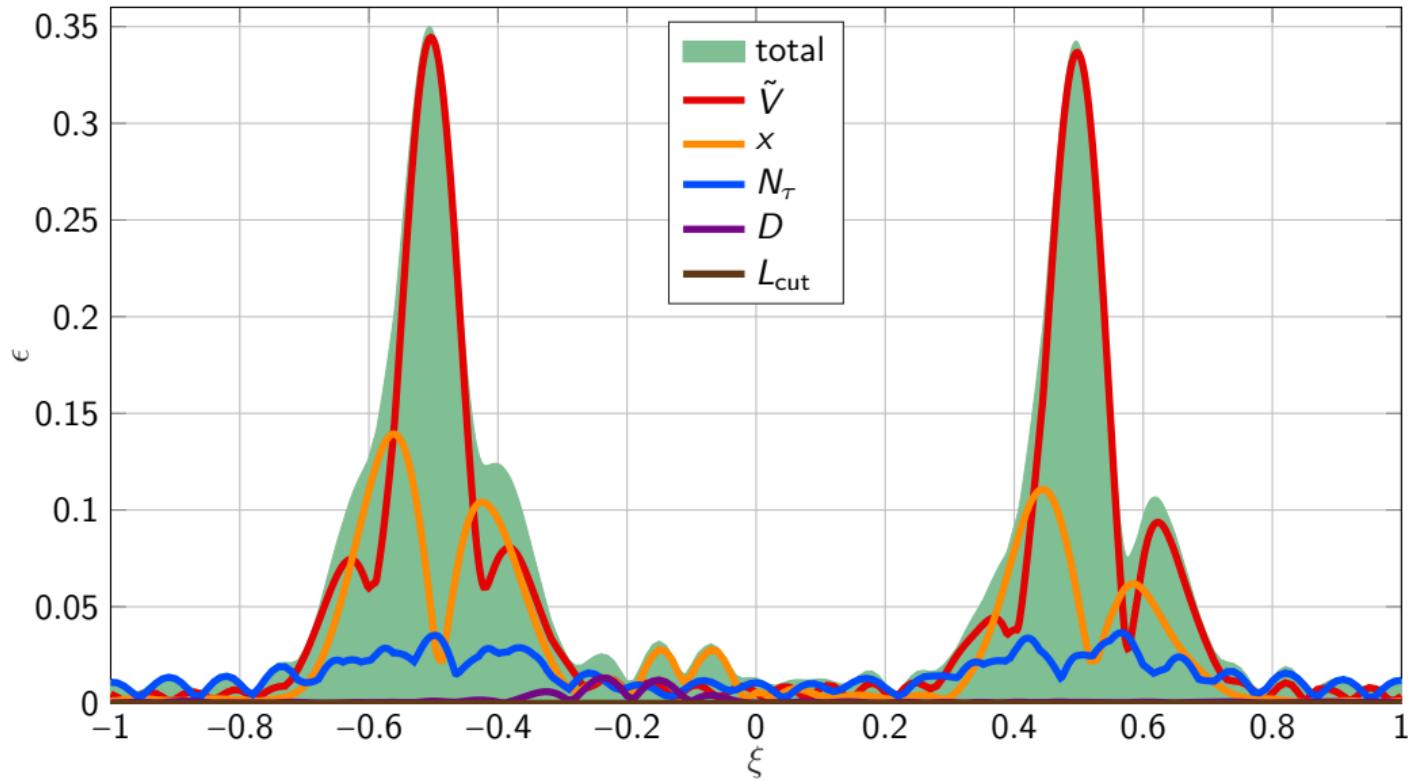
<sup>6</sup>J. Dai, J. Hughes, and J. Liu, “Perturbative analysis of the massless schwinger model,” Phys. Rev. D **51**, 5209–5215 (1995) doi:10.1103/PhysRevD.51.5209.

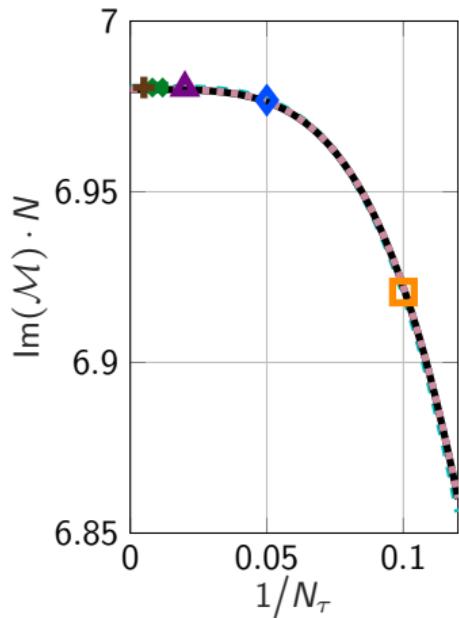
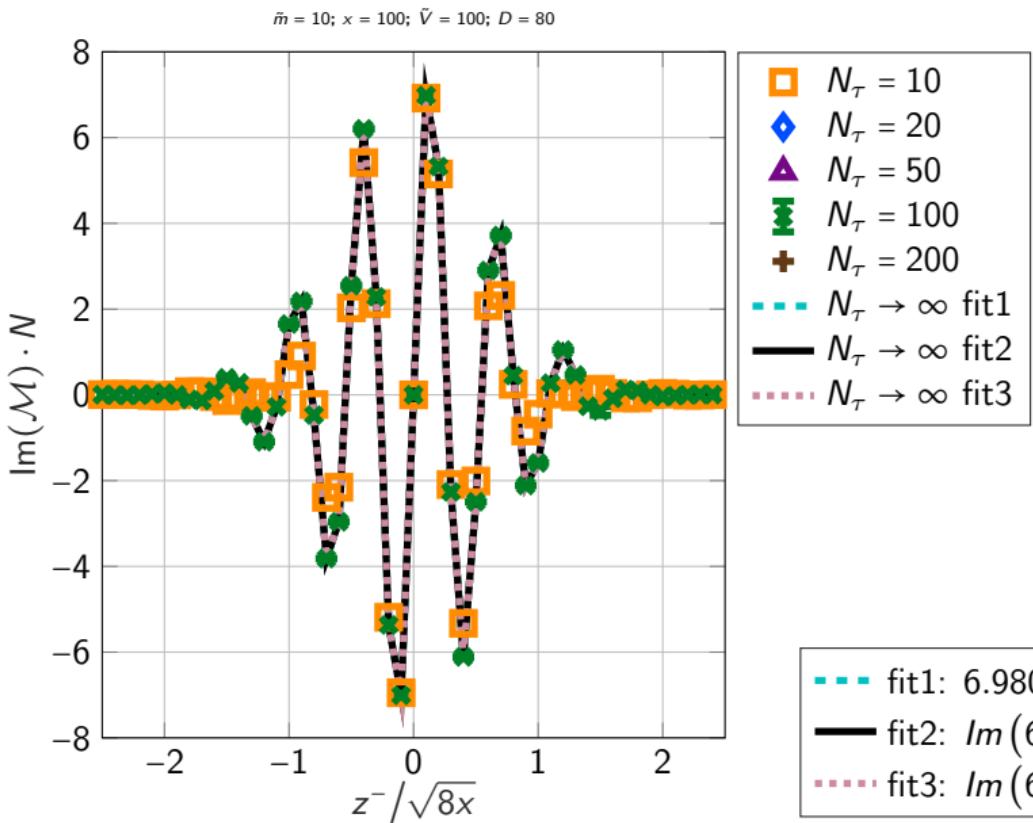
<sup>7</sup>C. J. Hamer, Z. Weihong, and J. Oitmaa, “Series expansions for the massive schwinger model in hamiltonian lattice theory,” Phys. Rev. D **56**, 55–67 (1997) doi:10.1103/PhysRevD.56.55.

<sup>8</sup>Further image sources, EIC, [www.computerhistory.org/timeline/1981](http://www.computerhistory.org/timeline/1981), <https://openmoji.org>, [www.flaticon.com/free-icons/search](http://www.flaticon.com/free-icons/search).

# Outline

## 6 Backup

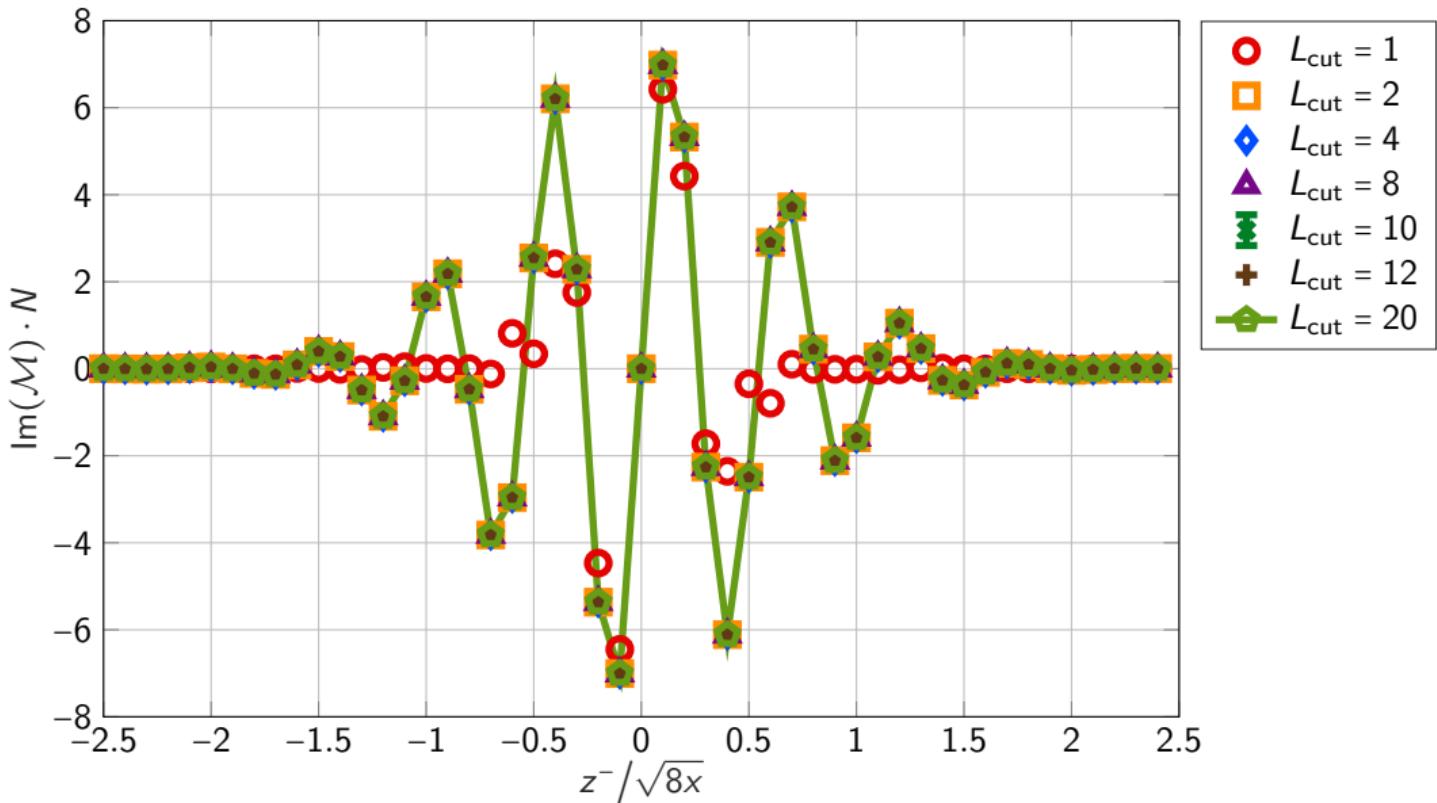
Contributions to error for  $\tilde{m} = 10$ 

Results:  $N_\tau$ -dependence

fit1:  $6.9805 + 0.0179N_\tau^{-2} - 600.9051N_\tau^{-4}$   
 fit2:  $\text{Im}(6.9800 \exp(-6.4460iN_\tau^{-2}d))$   
 fit3:  $\text{Im}(6.9800 \exp(-6.4411iN_\tau^{-2}d))$

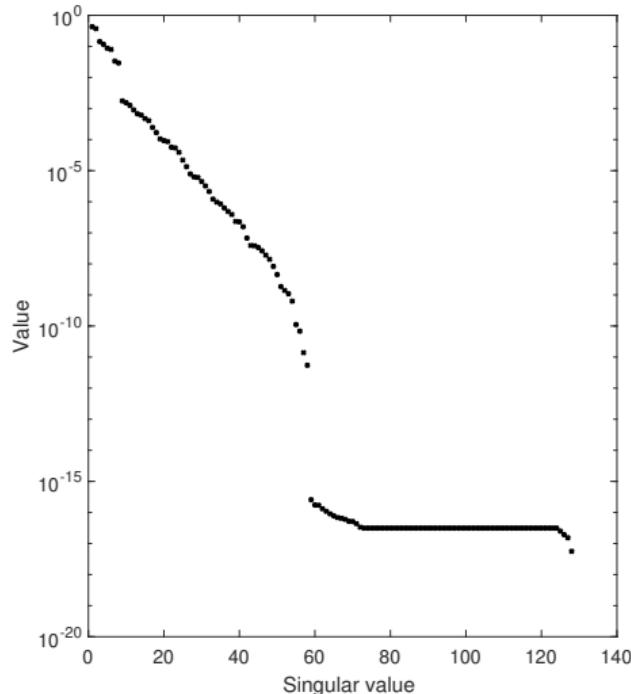
## $L_{\text{cut}}$ -dependence: truncation of electric field

$\tilde{m} = 10; x = 100; \tilde{V} = 100; D = 80; N_T = 100$

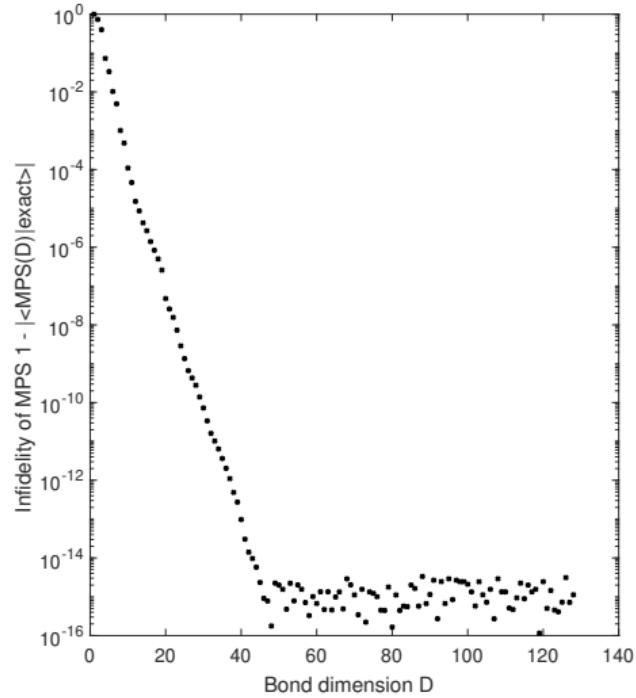


## Singular values and cutoff

Schwinger model,  $L = 14$ ,  $\mu = 0.125$ ,  $x = 10$ , 2nd excitation



Singular values for cut in the middle



Infidelity on MPS with exact state  
 $1 - |\langle \Psi(D) | \Psi_{\text{exact}} \rangle|$

## Efficient Tensor Network operations

- ▶ Find groundstate and excited states

$$\min \left( E = \frac{\langle \Psi | \hat{H} | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \begin{array}{c} \text{Diagram showing a tensor network for finding the groundstate energy. It consists of two horizontal layers of nodes. The top layer has 6 light blue circles connected by horizontal lines. The bottom layer has 6 green squares connected by horizontal lines. Vertical lines connect corresponding nodes between the two layers. The entire network is enclosed in large parentheses.} \end{array} \right)$$

- ▶ Apply operators / time evolution

$$\hat{O} |\Psi\rangle = \begin{array}{c} \text{Diagram showing the application of an operator } \hat{O} \text{ to a state } |\Psi\rangle. \text{ The state } |\Psi\rangle \text{ is represented by a tensor network with 6 light blue circles in the top layer and 6 green squares in the bottom layer, connected by vertical and horizontal lines. The operator } \hat{O} \text{ is represented by three green rectangles placed above the top layer. Dashed lines indicate the mapping from the state to the result.} \end{array} \rightarrow |\Phi\rangle = \begin{array}{c} \text{Diagram showing the resulting state } |\Phi\rangle. \text{ It consists of 6 pink circles connected by horizontal lines. Dashed lines map from the operator application diagram to this state.} \end{array}$$

- ▶ Calculate overlap

$$\langle \Psi | \Phi \rangle = \begin{array}{c} \text{Diagram showing the calculation of the overlap between states } |\Psi\rangle \text{ and } |\Phi\rangle. \text{ The left part shows the tensor network for } |\Psi\rangle \text{ (6 light blue circles). The right part shows the tensor network for } |\Phi\rangle \text{ (6 pink circles). They are connected by vertical lines, indicating the contraction of indices to calculate the inner product.} \end{array}$$

## Spin formulation of the Schwinger Model

$$\mathcal{L} = \bar{\Psi} (i\cancel{\partial} - g \cancel{A} - m) \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_0 \rho$$

$$\mathcal{H} = -i\bar{\Psi} \gamma^1 (\partial_1 - ig A_1) \Psi + m \bar{\Psi} \Psi + \frac{1}{2} E^2$$

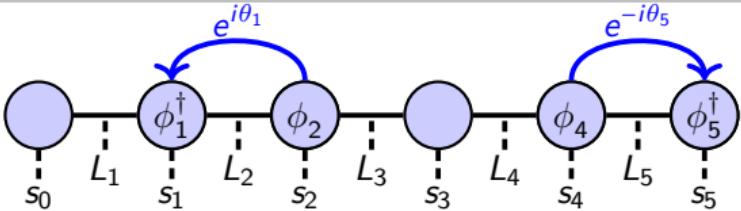
Legendre transformation  
( $E = F_{01}$ )

## Spin formulation of the Schwinger Model

$$\mathcal{L} = \bar{\Psi} (i\cancel{\partial} - g\cancel{A} - m) \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_0 \rho$$

$$\mathcal{H} = -i\bar{\Psi} \gamma^1 (\partial_1 - igA_1) \Psi + m\bar{\Psi} \Psi + \frac{1}{2} E^2$$

$$H = -\frac{i}{2a} \sum_n \left( \phi_n^\dagger e^{i\theta_n} \phi_{n+1} - \phi_{n+1}^\dagger e^{-i\theta_n} \phi_n \right) + m \sum_n (-1)^n \phi_n^\dagger \phi_n + \frac{ag^2}{2} \sum_n L_n^2$$



staggered fermions  
 $(\theta = agA_1, gL = E)$

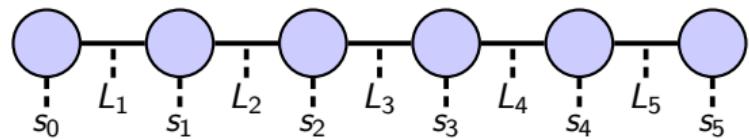
$$\phi_n \sim \begin{cases} \Psi_{\text{upper}}(x) & \text{if } n \text{ even} \\ \Psi_{\text{lower}}(x) & \text{if } n \text{ odd,} \end{cases}$$

## Spin formulation of the Schwinger Model

$$\mathcal{L} = \bar{\Psi} (i\cancel{\partial} - g \cancel{A} - m) \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_0 \rho$$

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decoupling

$$\phi_n \rightarrow \prod_{k < n} (e^{-i\theta_k}) \phi_n$$

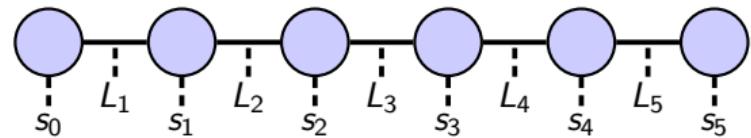
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$$\mathcal{L} = \bar{\Psi} (i\cancel{\partial} - g \cancel{A} - m) \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_0 \rho$$

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$$H = \frac{1}{2a} \sum_n (\sigma_n^+ \sigma_{n+1}^- + \sigma_{n+1}^- \sigma_n^+) + \frac{m}{2} \sum_n [1 + (-1)^n \sigma_n^z] + \frac{ag^2}{2} \sum_n L_n^2$$



Jordan-Wigner  
transformation

$$\hat{\phi}_n = \prod_{k < n} (i\sigma_k^z) \sigma_n^-$$

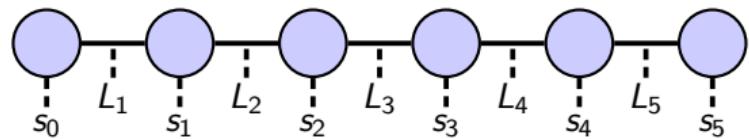
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$$\mathcal{L} = \bar{\Psi} (i\cancel{\partial} - g \cancel{A} - m) \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_0 \rho$$

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$$H = \frac{1}{2a} \sum_n (\sigma_n^+ \sigma_{n+1}^- + \sigma_{n+1}^- \sigma_n^+) + \frac{m}{2} \sum_n [1 + (-1)^n \sigma_n^z] + \frac{ag^2}{2} \sum_n L_n^2$$



Gauss's law:  
 $(x = \frac{1}{a^2 g^2}, \mu = \frac{2m}{ag^2})$

$$L_n - L_{n-1} = Q_n = \frac{1}{2} [(-1)^n + \sigma_n^z] + q_n$$

## Spin formulation of the Schwinger Model

$$\mathcal{L} = \bar{\Psi} (i\cancel{\partial} - g \cancel{A} - m) \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_0 \rho$$

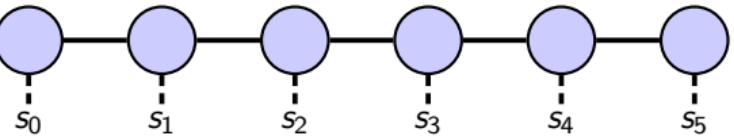
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$$H = -\frac{i}{2a} \sum_n \left( \phi_n^\dagger e^{i\theta_n} \phi_{n+1} - \phi_{n+1}^\dagger e^{-i\theta_n} \phi_n \right) + m \sum_n (-1)^n \phi_n^\dagger \phi_n + \frac{ag^2}{2} \sum_n L_n^2$$

$$H = \frac{1}{2a} \sum_n (\sigma_n^+ \sigma_{n+1}^- + \sigma_{n+1}^- \sigma_n^+) + \frac{m}{2} \sum_n [1 + (-1)^n \sigma_n^z] + \frac{ag^2}{2} \sum_n L_n^2$$

$$H = x \sum_{n=0}^{N-2} [\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+] + \frac{\mu}{2} \sum_{n=0}^{N-1} [1 + (-1)^n \sigma_n^z] + \sum_{n=0}^{N-2} \left[ \sum_{k=0}^n Q_k \right]^2$$

Gauss's law:  
 $(x = \frac{1}{a^2 g^2}, \mu = \frac{2m}{ag^2})$



$$L_n - L_{n-1} = Q_n = \frac{1}{2} [(-1)^n + \sigma_n^z] + q_n$$

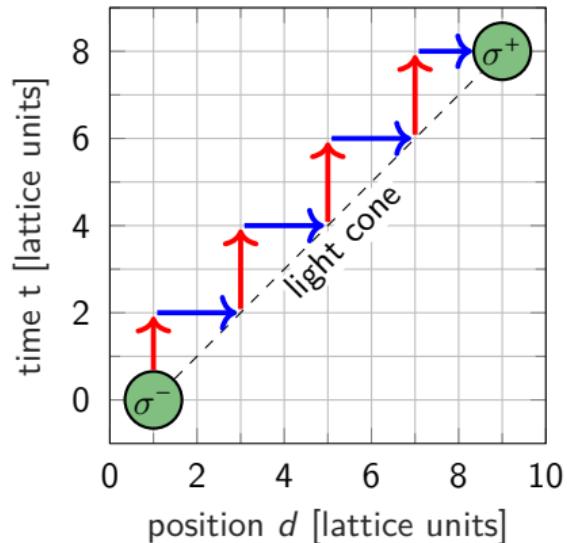
## Lattice formulation of PDF

PDF for lattice spin model:

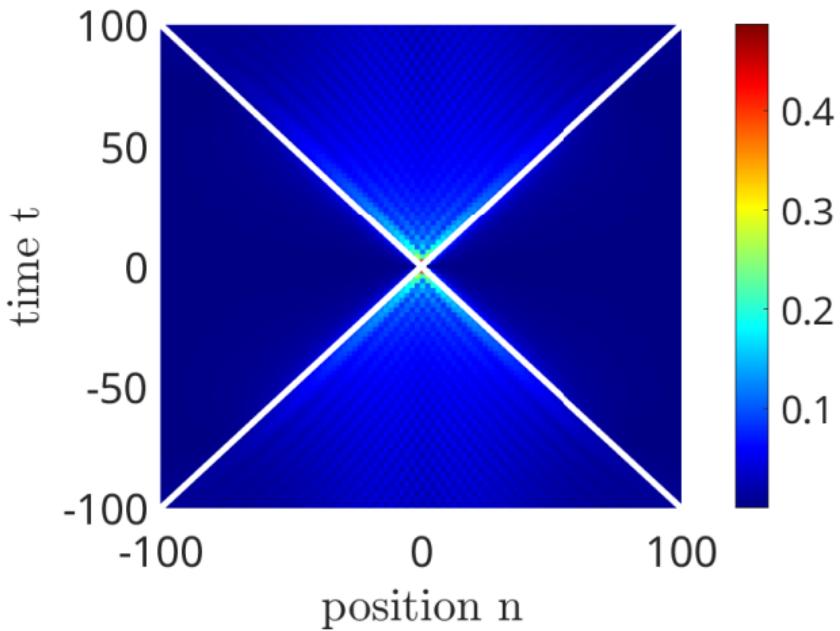
$$f_{\Psi}(\xi) = \frac{NM}{8\pi x} \sum_{d=0,2,4,\dots,L} e^{-i\xi \frac{Md}{2x}} \mathcal{M}(d)$$

$$\mathcal{M}(d) = \mathcal{M}_{0,0}(d) - \mathcal{M}_{0,1}(d) - \mathcal{M}_{1,0}(d) + \mathcal{M}_{1,1}(d)$$

$$\begin{aligned} \mathcal{M}_{a,b}(d) = & \langle h | e^{iHt_d} \prod_{k < d+a} (-i\sigma_k^z) \sigma_{d+a}^+ e^{-iH_{d-1}\delta t} \\ & \dots e^{-iH_3\delta t} e^{-iH_1\delta t} \prod_{k' < b} (i\sigma_{k'}^z) \sigma_b^- | h \rangle_c. \end{aligned}$$



## Light-cone structure



$$\langle P | e^{iHt} \prod_{k < n} (i\sigma_k^z) \sigma_n^+ e^{-iH_0 t} \prod_{k' < 0} (-i\sigma_{k'}^z) \sigma_0^- | P \rangle$$

- ▶ even-to-even matrix element
- ▶ calculated to each site at each timeslice
- ▶ static charge fixed at origin