Geometrical View of Quantum Phase Transition: From Kähler to Pseudo-Kähler Structure Transition

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Introduction

◉ Quantum Phase transition (QPT) at T=0 S. Sachdev *"Quantum Phase Transition"*

With a continuous change of system parameters, the quantum ground state undergoes a nonanalytic change, particularly in its quantum phase structure.

$$\hat{H}(\boldsymbol{k})|\psi_g(\boldsymbol{k})\rangle = E_g(\boldsymbol{k})|\psi_g(\boldsymbol{k})\rangle$$ ($\boldsymbol{k}$ system parameter)

Ground state : $|\psi_g(\boldsymbol{k})\rangle = \boxed{e^{i\eta(\boldsymbol{k})}}|\psi'_g(\boldsymbol{k})\rangle$

quantum phase factor (physically same)

[Berry 1984]

・Berry phase $\eta_g := \oint_C \underline{A_\mu} dk^\mu = (-i)\oint_C \underline{\langle\psi_g(\boldsymbol{k})|\partial_\mu\psi_g(\boldsymbol{k})\rangle} dk^\mu$

(topological invariant : holonomy) Berry connection

[Carollo, *et al.*, 2005]

・ Application to QPT
- ・1D XY spin chain
- ・superradiance phase transition
- ・Topological materials

Singularity of the Berry phase

Geometric structure of a pure quantum state

- **Principal fiber bundle** $P(B,G)$

B. Simon, PRL **51**, 2167 (1983)

Physical projective space $[\psi(\boldsymbol{k})] \sim e^{i\eta(\boldsymbol{k})}|\psi(\boldsymbol{k})\rangle$

To define a path across the fibers :

$$\boxed{\text{Berry connection } A_\mu(\boldsymbol{k}) := -i\langle\psi_g(\boldsymbol{k})|\partial_\mu\psi_g(\boldsymbol{k})\rangle}$$

Geometrical phase (topological invariant, global)

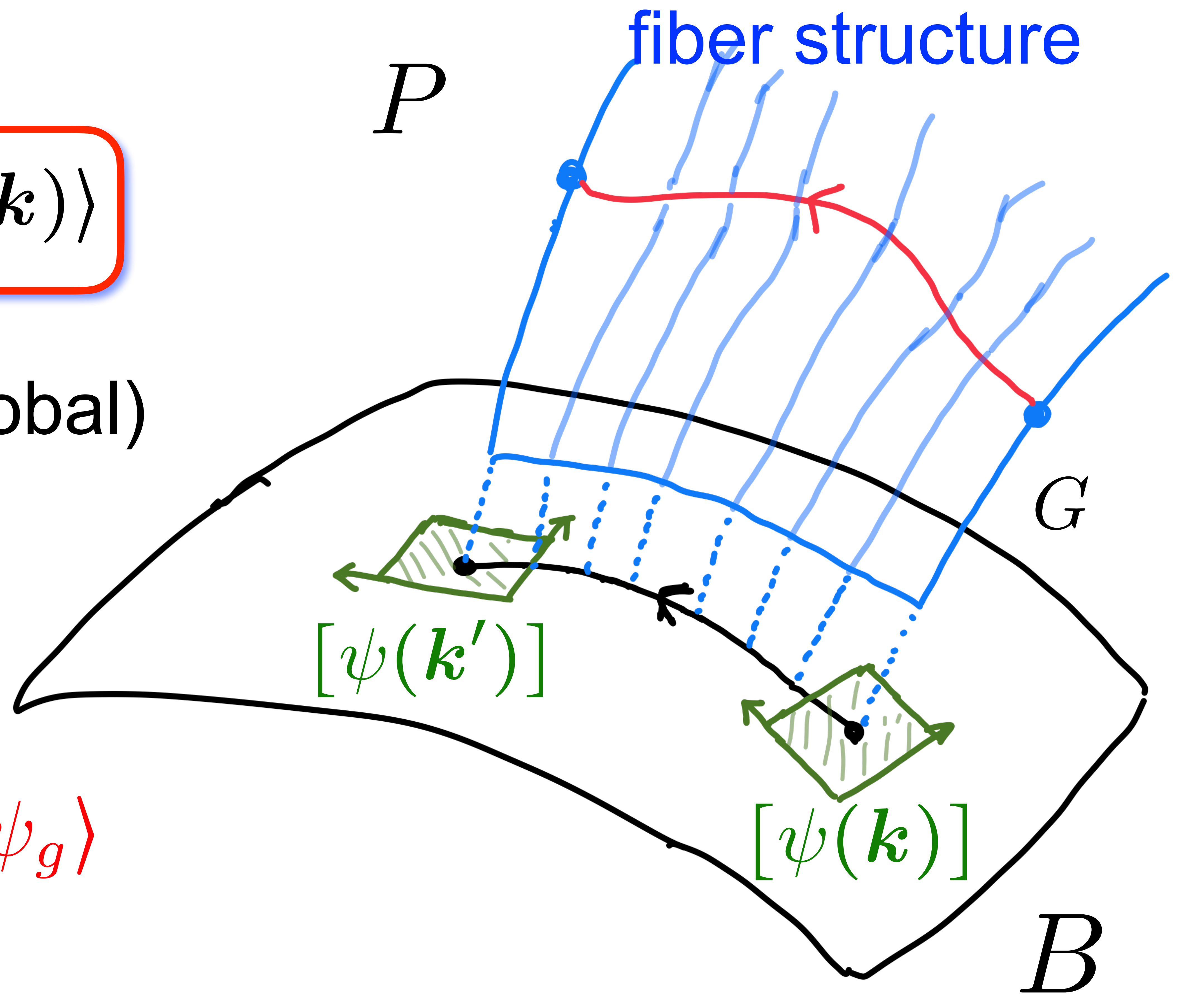
$$\eta_g = \oint_C A_\mu dk^\mu = \int_S F_{\mu\nu} dk^\mu \wedge dk^\nu$$

Quantum geometric tensor (local field)

$$Q_{\mu\nu}(\boldsymbol{k}) = \langle\partial_\mu\psi_g|\partial_\nu\psi_g\rangle - \langle\partial_\mu\psi_g|\psi_g\rangle\langle\psi_g|\partial_\nu\psi_g\rangle$$

(Zanardi 2006)

Ground state in the Hilbert space $|\psi_g\rangle \in \mathcal{H}$



projective Hilbert space

◉ Bosonic QPT system

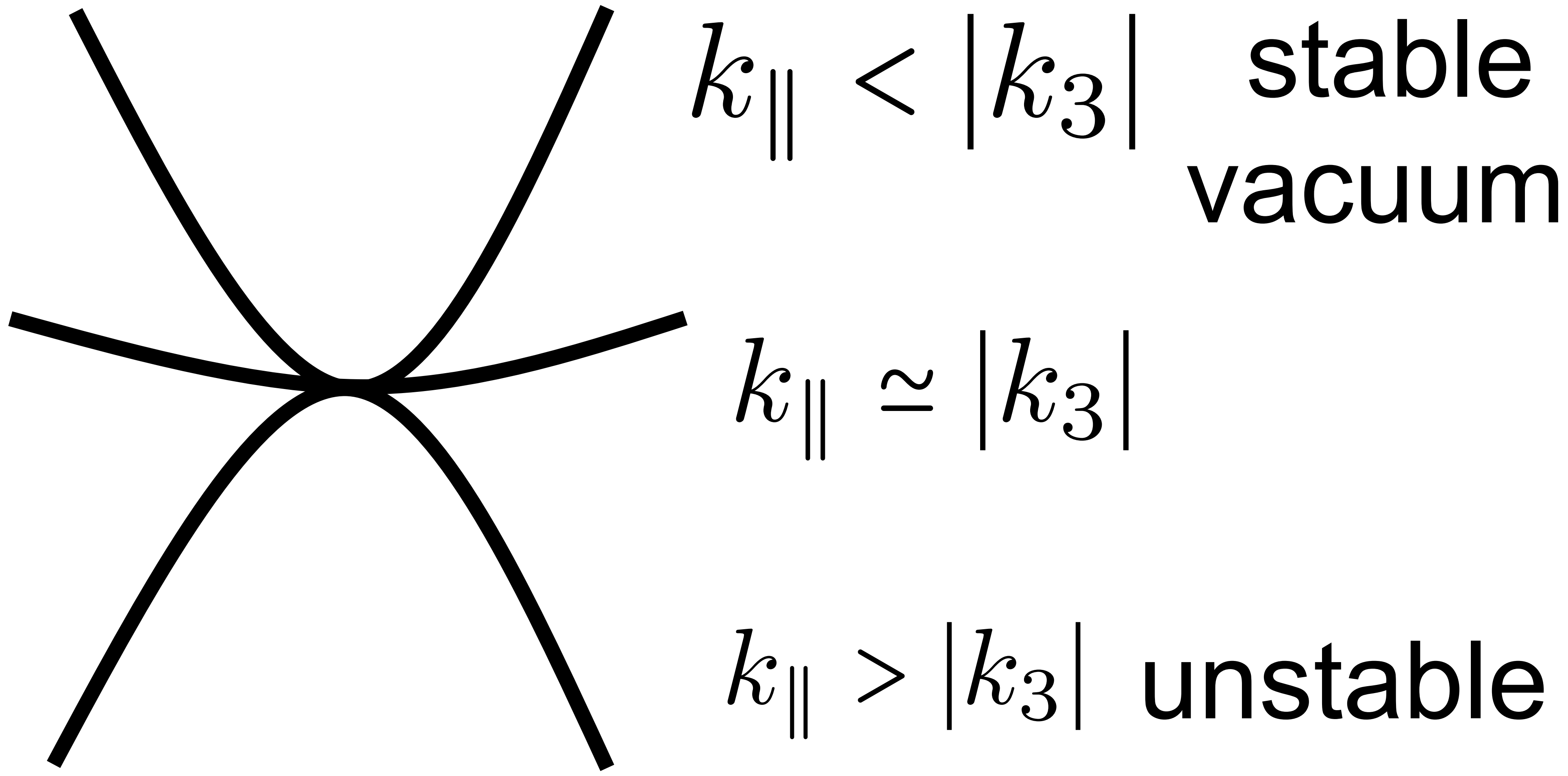
$$\hat{H}(\boldsymbol{k}) = \frac{k_3}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) + \frac{k_\parallel}{2}\left(e^{-i\theta}\hat{a}^{\dagger 2} + e^{i\theta}\hat{a}^2\right)$$

Ground (vacuum) state virtual transition

$$|\psi_g(\boldsymbol{k})\rangle = \sum_{n=0}^{\infty} c_n(\boldsymbol{k})|n\rangle$$

divergent series

$\notin \mathcal{H}$



e.g. Parametric amplification
instability of quantum vacuum

◉ Problem

- Berry connection based on Hilbert space inner product is inapplicable.
- A new definition of the connection is needed for a bosonic QPT system.

◉ Purpose

- A new definition of connection (QGT) for the entire parameter space.
- Drawing a geometric structure based on QGT: Kähler => pseudo-Kähler structure

Model

General form of a single boson quadratic coupling

Hermitian $\hat{H}(\boldsymbol{k}) = k_3\hat{K}_3 + \underline{k_1\hat{K}_1 + k_2\hat{K}_2} \in \mathfrak{su}(1,1)$: non-compact group

virtual transition

・Generators $\hat{K}_3 = \frac{1}{2}\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right)$, $\underline{\hat{K}_1 = -i\frac{1}{2}\left(\hat{a}^2 - \hat{a}^{\dagger 2}\right) , \hat{K}_2 = \frac{1}{2}\left(\hat{a}^2 + \hat{a}^{\dagger 2}\right)}$

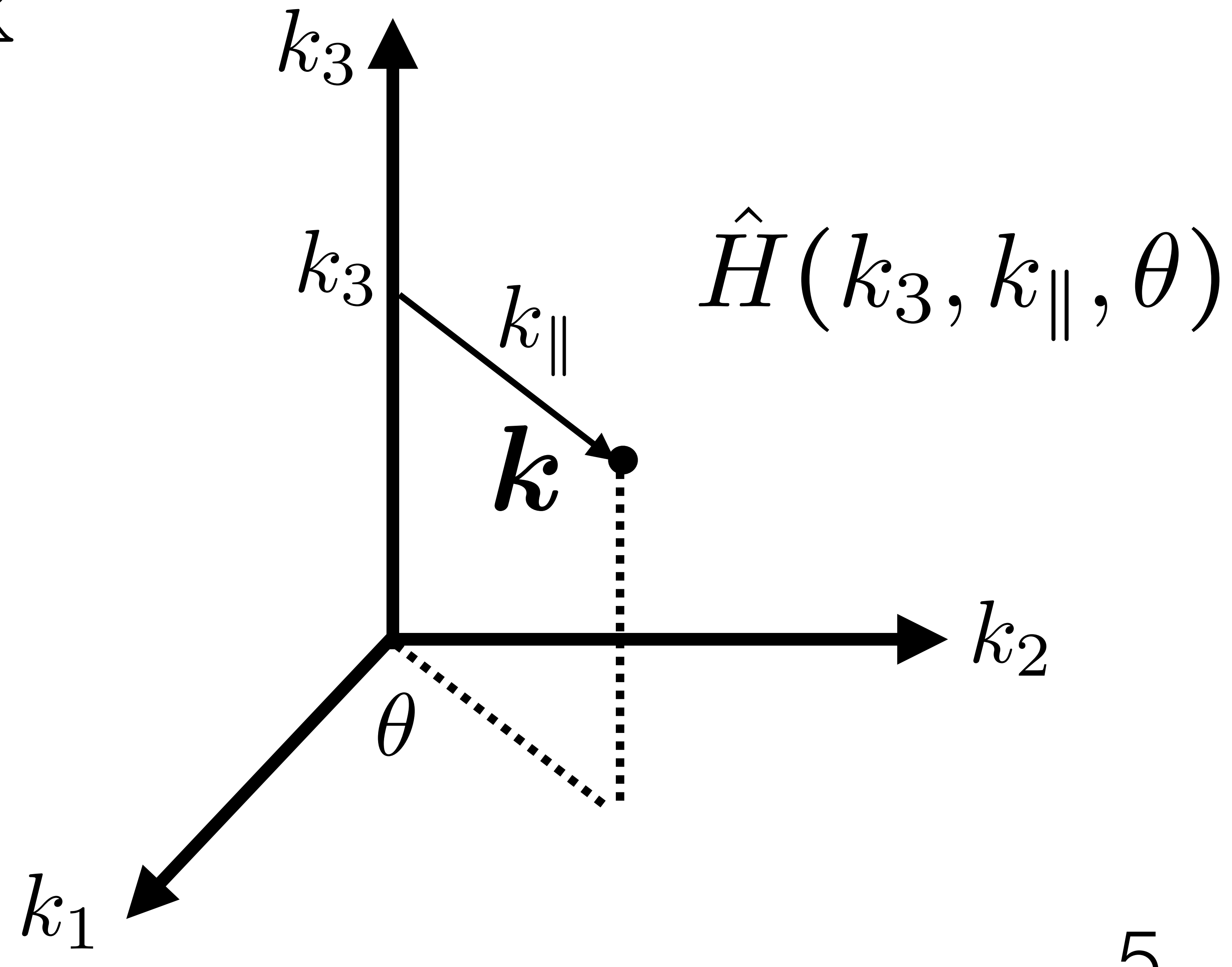
$\mathfrak{su}(1,1)$ Lie algebra : $\left[\frac{\hat{K}_1}{2}, \frac{\hat{K}_2}{2}\right] = -i\frac{\hat{K}_3}{2}$, $\left[\frac{\hat{K}_2}{2}, \frac{\hat{K}_3}{2}\right] = i\frac{\hat{K}_1}{2}$, $\left[\frac{\hat{K}_3}{2}, \frac{\hat{K}_1}{2}\right] = i\frac{\hat{K}_2}{2}$

Parameter space : $\boldsymbol{k} = (k_\mu) = (k_1, k_2, k_3) = (k_\parallel, \theta, k_3) \in \mathbb{R}^3$

cylindrical coordinates $k_1 = -k_\parallel \sin\theta$, $k_2 = k_\parallel \cos\theta$

unperturbed frequency k_3

perturbation strength $k_\parallel$ phase θ

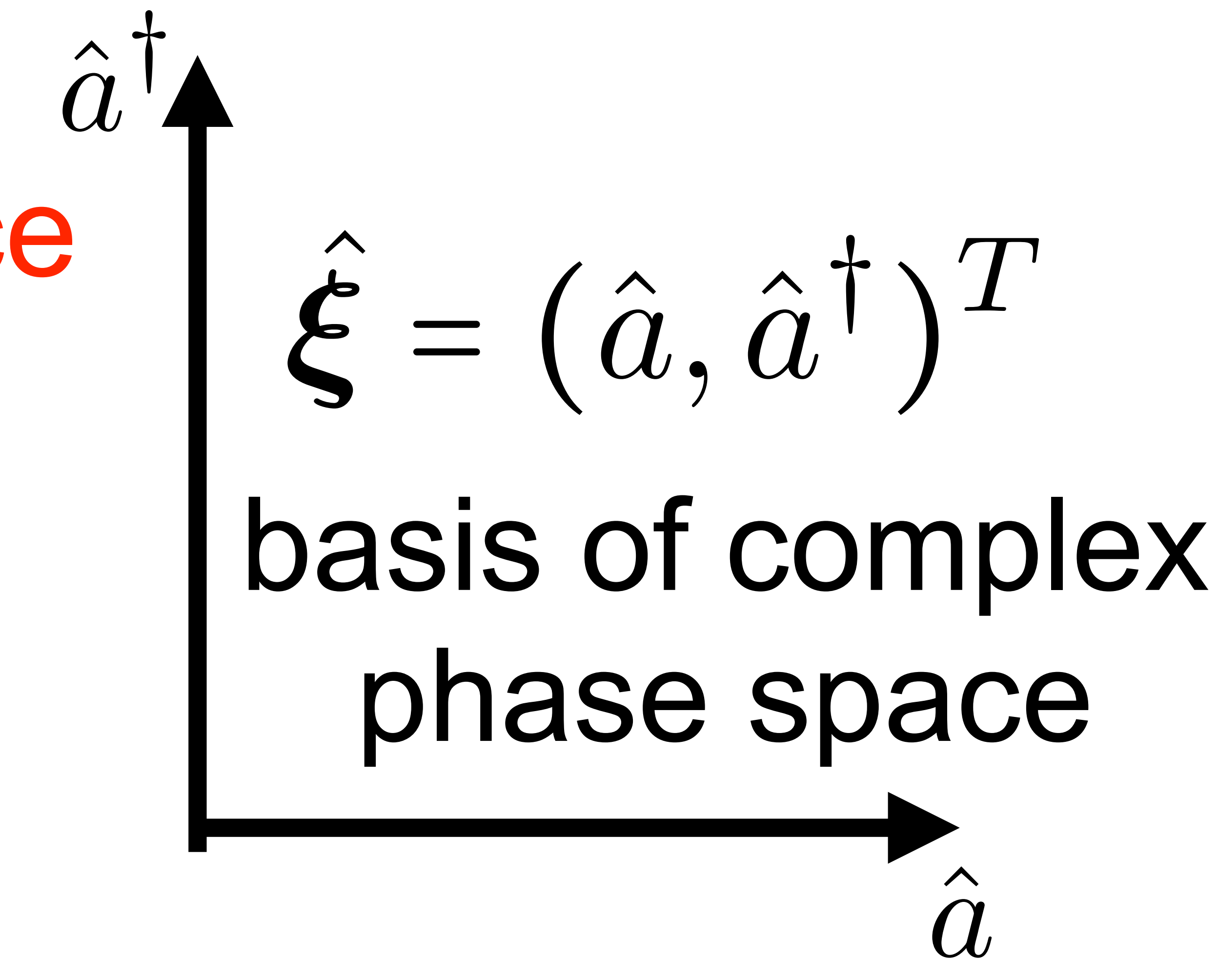


We obtain the eigenmodes $\{\hat{\varphi}(\boldsymbol{k}), \hat{\varphi}^\star(\boldsymbol{k})\}$

=> QGT corresponding to $\{\hat{\varphi}(\boldsymbol{k}), \hat{\varphi}^\star(\boldsymbol{k})\}$

Complex eigenmodes across the parameter space

Representation in terms of the complex phase (symplectic) space



$$\hat{H}(\boldsymbol{k}) = \frac{1}{2}(\hat{a}, \hat{a}^\dagger)\begin{pmatrix} k_\parallel e^{i\theta} & k_3 \\ k_3 & k_\parallel e^{-i\theta} \end{pmatrix}\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \frac{1}{2}\hat{\boldsymbol{\xi}}^T \boldsymbol{H}(\boldsymbol{k})\hat{\boldsymbol{\xi}} \,,$$

Hamiltonian matrix

Basis of the complex phase space: $\hat{\boldsymbol{\xi}} = (\hat{a}, \hat{a}^\dagger)^T$

$\boldsymbol{\Omega}$ symplectic form

Symplectic form is represented by $[\hat{a}, \hat{a}^\dagger] = (\hat{a}, \hat{a}^\dagger)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = 1$

C. Weedbrook *et al.*, RMP **84**, 621(2012)

Heisenberg eq. $i\frac{d}{dt}\hat{\boldsymbol{\xi}}(t) = \boldsymbol{\Omega}\boldsymbol{H}(\boldsymbol{k})\hat{\boldsymbol{\xi}}(t) = \boldsymbol{L}(\boldsymbol{k})\hat{\boldsymbol{\xi}}(t) \; (= [\hat{\boldsymbol{\xi}}(t), \hat{H}(\boldsymbol{k})])$

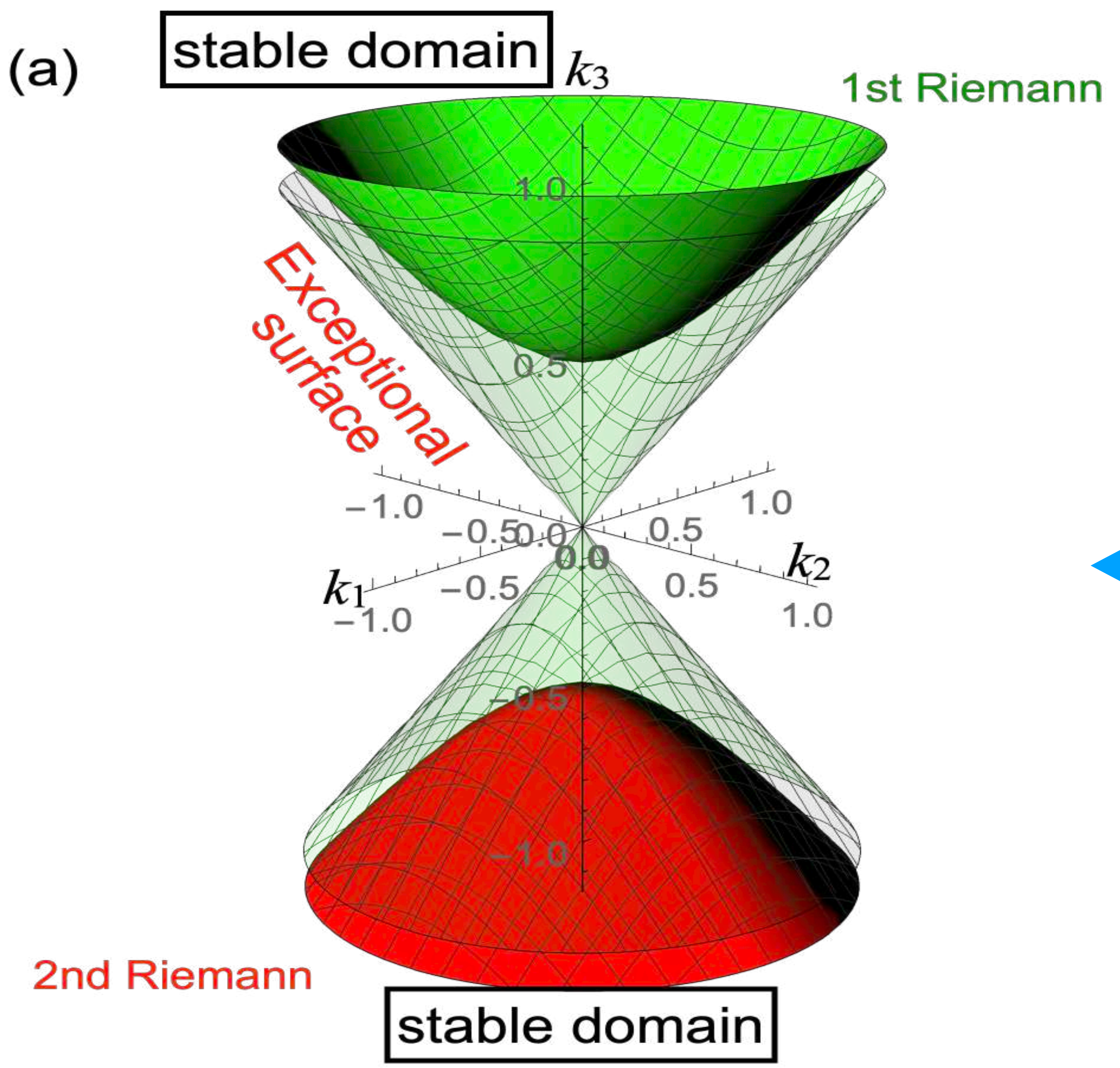
Liouvillian $\boldsymbol{L}(\boldsymbol{k}) = \boldsymbol{\Omega}\boldsymbol{H}(\boldsymbol{k}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} k_\parallel e^{i\theta} & k_3 \\ k_3 & k_\parallel e^{-i\theta} \end{pmatrix} = \begin{pmatrix} k_3 & k_\parallel e^{-i\theta} \\ -k_\parallel e^{i\theta} & -k_3 \end{pmatrix}$

Nonhermitian

- Symplectic symmetry $\boldsymbol{\Omega L \Omega} = \boldsymbol{L}^T$
- Diagonalization of $\boldsymbol{L}$ $\tilde{\boldsymbol{\Phi}}_\alpha(\boldsymbol{k})\boldsymbol{L}(\boldsymbol{k})\boldsymbol{\Phi}_\alpha(\boldsymbol{k}) = \boldsymbol{Z}_\alpha(\boldsymbol{k}) = \begin{pmatrix} z_\alpha(\boldsymbol{k}) & 0 \\ 0 & -z_\alpha(\boldsymbol{k}) \end{pmatrix}$

symplectic $\tilde{\boldsymbol{\Phi}}_\alpha^T \boldsymbol{\Omega} \tilde{\boldsymbol{\Phi}}_\alpha = \boldsymbol{\Omega}$, $\boldsymbol{\Phi}_\alpha^T \boldsymbol{\Omega} \boldsymbol{\Phi}_\alpha = \boldsymbol{\Omega}$, $\tilde{\boldsymbol{\Phi}}_\alpha \boldsymbol{\Phi}_\alpha = \boldsymbol{I}$

$\alpha = 1, 2$: index of Riemann sheets



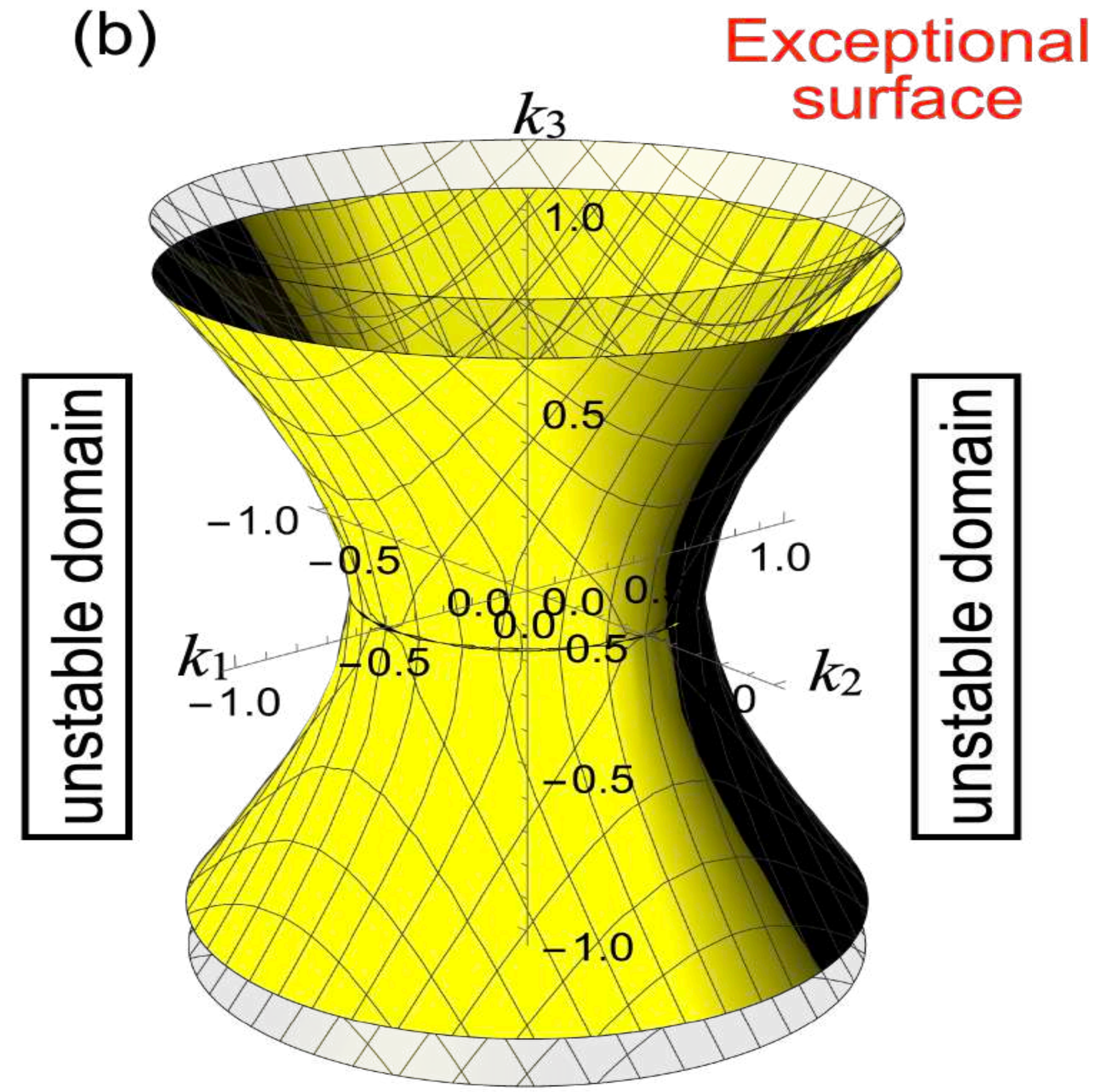
Equienergy surface

$$z_\alpha(\boldsymbol{k}) = (k_3^2 - k_\parallel^2)^{1/2}$$

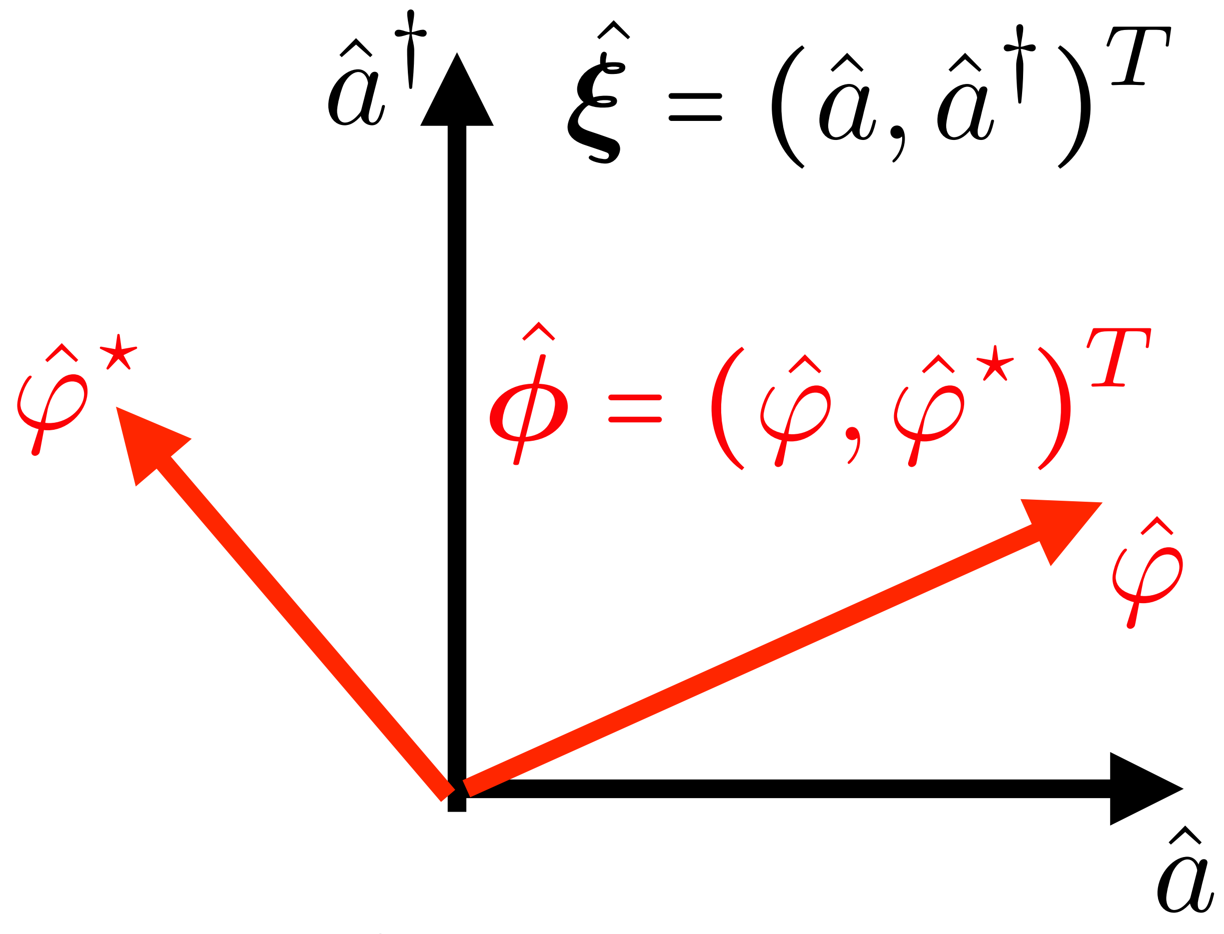
phase boundary $|k_3| = k_\parallel$

$\mathrm{Re}[z(\boldsymbol{k})] \neq 0$

$\mathrm{Im}[z(\boldsymbol{k})] \neq 0$



◉ Transformation bare to the "eigenmodes"



$$\hat{\boldsymbol{\phi}}(\boldsymbol{k}) = \begin{pmatrix} \hat{\varphi}_\alpha(\boldsymbol{k}) \\ \hat{\varphi}^\star_\alpha(\boldsymbol{k}) \end{pmatrix} = \tilde{\boldsymbol{\Phi}}_\alpha(\boldsymbol{k}) \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

・Analytic symplectic transformation

$$\boldsymbol{\phi}_\alpha(\boldsymbol{k}) = \begin{pmatrix} \hat{\varphi}_\alpha(\boldsymbol{k}) \\ \hat{\varphi}^\star_\alpha(\boldsymbol{k}) \end{pmatrix} = \tilde{\boldsymbol{\Phi}}_\alpha(\boldsymbol{k}) \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \begin{pmatrix} e^{\boxed{i\eta_\alpha(\boldsymbol{k})}} & 0 \\ 0 & e^{\boxed{-i\eta_\alpha(\boldsymbol{k})}} \end{pmatrix} \begin{pmatrix} e^{\frac{i\theta}{2}} \cosh\beta_\alpha & e^{\frac{-i\theta}{2}} \sinh\beta_\alpha \\ e^{\frac{i\theta}{2}} \sinh\beta_\alpha & e^{\frac{-i\theta}{2}} \cosh\beta_\alpha \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$$

gauge freedom

hyperboloid angle $\beta_\alpha = \frac{1}{4} \ln\left(\frac{k_3 + k_\parallel}{k_3 - k_\parallel} \right)$

$$\begin{cases} \beta_\alpha \in \mathbb{R}\,, & |k_3| > k_\parallel \quad \text{stable} \\ \beta_\alpha \in \mathbb{C}\,, & |k_3| < k_\parallel \quad \text{unstable} \end{cases}$$

$\hat{\varphi}^\star_\alpha(\boldsymbol{k}') \neq \hat{\varphi}^\dagger_\alpha(\boldsymbol{k})$ (unstable) $\quad [\hat{\varphi}_\alpha(\boldsymbol{k}), \hat{\varphi}^\star_\alpha(\boldsymbol{k})] = 1$ always satisfied

◉ Diagonalization of the Hamiltonian $\hat{H}(\boldsymbol{k}) = \frac{z_\alpha(\boldsymbol{k})}{2} \left(\hat{\varphi}^\star_\alpha(\boldsymbol{k}) \hat{\varphi}_\alpha(\boldsymbol{k}) + \hat{\varphi}_\alpha(\boldsymbol{k}) \hat{\varphi}^\star_\alpha(\boldsymbol{k}) \right)$

(Hermitian with complex eigenvalues in unstable region)

Geometrical structure : principal bundle with U(1) structure group

Local coordinates on B
$\boldsymbol{\lambda} = (\lambda^\mu) = (\theta, \beta)$

- Principal bundle manifold $P(B, G)$
- Operator projective space $[\phi_\alpha(\boldsymbol{\lambda})] \sim \underbrace{\exp[i\boldsymbol{\eta}_\alpha(\boldsymbol{\lambda})]}_{g(\boldsymbol{\lambda})}\phi_\alpha(\boldsymbol{\lambda})$

Definition of the Berry connection in the bundle theory

- Ehresmann connection $T_u(P) = V_u(P) \oplus H_u(P)$

  Horizontal space $H_u(P) = \{X_u \in T_u P \,|\, \omega(X_u) = 0\}$

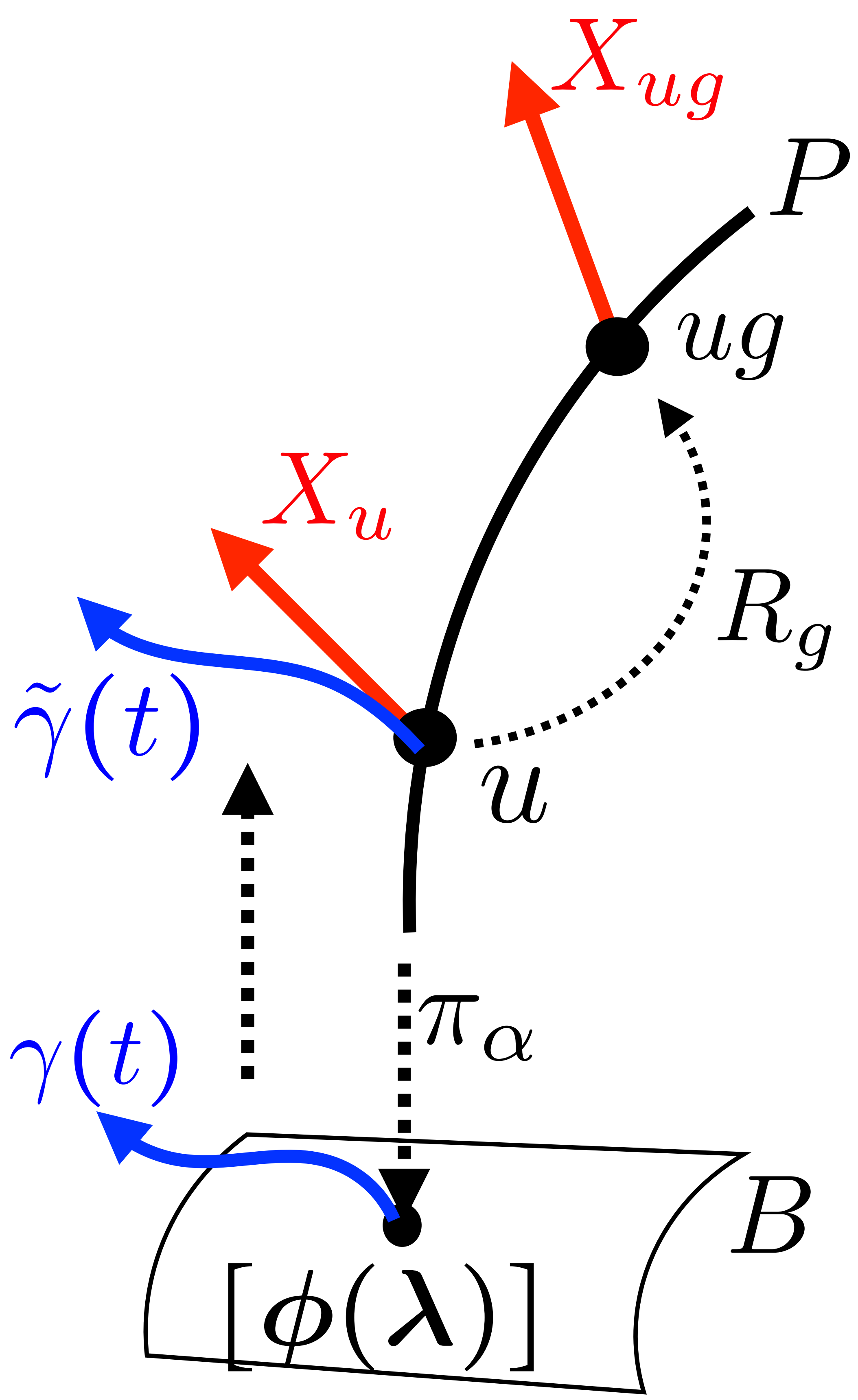
- Local $\mathfrak{g}$-valued connection 1-form on P $\omega \in \mathfrak{g} \otimes T^*(P)$

$$\omega_\alpha = g_\alpha^{-1} \pi_\alpha^* \boldsymbol{A}_\alpha g_\alpha + g_\alpha^{-1} d_P g_\alpha \qquad (\boldsymbol{A}_\alpha \in \mathfrak{g} \otimes T^*(B))$$

$$\boldsymbol{A}_\alpha = \boxed{A_{\alpha,\mu}(\boldsymbol{\lambda})} d\lambda^\mu$$ Berry connection

$$\boxed{\text{We take } A_\mu(\boldsymbol{\lambda}) = -i[\varphi_e(\boldsymbol{\lambda}), \partial_\mu \varphi_e^\star(\boldsymbol{\lambda})] \text{ on } B}$$

e:standard local trivialization



- Horizontal lift of the curve $\gamma(t) = (\lambda^\mu(t)) \longrightarrow \tilde{\gamma}(t) = (\lambda^\mu(t), g(t)) \in H_u(P)$

Covariant derivative

$$\frac{d}{dt}\tilde{\gamma}(t) = \frac{d\lambda^\mu}{dt}(t) D_\mu^{(P)} \tilde{\gamma}$$

・Covariant derivative on B

$$D_\mu^{(B)} = \partial_\mu + iA_\mu(\boldsymbol{\lambda}) = \partial_\mu + [\varphi(\boldsymbol{\lambda}), \partial_\mu \varphi^\star(\boldsymbol{\lambda})]$$

$$D_\mu^{(B)\star} = \partial_\mu - iA_\mu(\boldsymbol{\lambda}) = \partial_\mu - [\varphi(\boldsymbol{\lambda}), \partial_\mu \varphi^\star(\boldsymbol{\lambda})]$$

defines the tangent vector on the projective space

$$[\varphi, D_\mu^{(B)\star} \varphi^\star] = [D_\mu^{(B)} \varphi, \varphi^\star] = 0$$

$$\Longrightarrow \quad [\varphi(\boldsymbol{\lambda} + d\boldsymbol{\lambda}), \varphi^\star(\boldsymbol{\lambda} + d\boldsymbol{\lambda})] = [\varphi(\boldsymbol{\lambda}), \varphi^\star(\boldsymbol{\lambda})] = 1$$

Conservation of the canonical commutation relation

Berry connection and covariant derivatives are defined by the operator commutation relations.

Quantum geometric tensor (QGT): Local measure of the quantum geometry

(0,2)-type tensor on the projective manifold ***B***

Geometrical relation between the two tangent vectors of eigenmodes on ***B***

$$Q^{(B)}_{\mu\nu}(\boldsymbol{\lambda}) := [\underline{D^{(B)\star}_{\mu}\varphi^{\star}(\boldsymbol{\lambda})}, \underline{D^{(B)}_{\nu}\varphi(\boldsymbol{\lambda})}] = [\partial_\mu\varphi^\star, \partial_\nu\varphi] - [\partial_\mu\varphi^\star, \varphi][\partial_\nu\varphi, \varphi^\star]$$

non-commutativity => holonomy (Berry curvature)

$$(\ cf.\ \ Q_{\mu\nu}(\boldsymbol{\lambda}) = \langle\partial_\mu\psi_g|\partial_\nu\psi_g\rangle - \langle\partial_\mu\psi_g|\psi_g\rangle\langle\psi_g|\partial_\nu\psi_g\rangle)$$

$$Q^{(B)}_{\mu\nu} = \begin{array}{cc} d\theta & d\beta \\ \left(\begin{matrix} \frac{1}{4}\sinh^2 2\beta & -\frac{i}{2}\sinh 2\beta \\ \frac{i}{2}\sinh 2\beta & 1 \end{matrix}\right. & \left.\right) \end{array} = \begin{array}{cc} d\theta & d\beta \\ \left(\begin{matrix} \frac{1}{4}\frac{k_\parallel^2}{k_3^2-k_\parallel^2} & -\frac{i}{2}\frac{k_\parallel}{\sqrt{k_3^2-k_\parallel^2}} \\ \frac{i}{2}\frac{k_\parallel}{\sqrt{k_3^2-k_\parallel^2}} & 1 \end{matrix}\right. & \left.\right) \end{array}$$

$$\tanh 2\beta_\alpha = \frac{k_\parallel}{k_3}\ ,\ \tan\theta = \frac{k_2}{k_1}$$

QGT across QPT

$$Q_{\mu\nu} = g_{\mu\nu} + \frac{i}{2}F_{\mu\nu}$$

$|k_3| = k_\parallel$ divergence at QPT

◉ Symmetric part : Quantum metric

$$g_{\mu\nu} := \frac{1}{2}(Q_{\mu\nu} + Q_{\nu\mu}) = \begin{array}{cc} d\theta & d\beta \end{array} \\ \begin{pmatrix} \frac{1}{4}\frac{k_\parallel^2}{\boxed{k_3^2-k_\parallel^2}} & 0 \\ 0 & 1 \end{pmatrix}$$

positive => partially negative

Riemann => pseudo Riemann

・Line element :Fubini-Study distance

$$ds^2 = g_{\mu\nu}d\lambda^\mu d\lambda^\nu = \left(\frac{1}{4}\frac{k_\parallel^2}{k_3^2 - k_\parallel^2}\right)d\theta^2 + d\beta^2$$

◉ Antisymmetric part : Berry curvature

$$F_{\mu\nu} := -i(Q_{\mu\nu} - Q_{\nu\mu}) = \begin{array}{cc} d\theta & d\beta \end{array} \\ \begin{pmatrix} 0 & -\frac{1}{2}\frac{k_\parallel}{\boxed{\sqrt{k_3^2-k_\parallel^2}}} \\ \frac{1}{2}\frac{k_\parallel}{\boxed{\sqrt{k_3^2-k_\parallel^2}}} & 0 \end{pmatrix}$$

real phase => complex phase

・Berry phase (topological invariant) $\Gamma_g = \int_S F_{\mu\nu} d\lambda^\mu \wedge d\lambda^\nu$

◉ Kähler structure …. compatibility of the three structures (Ashteker 1999)

inherit to a pure quantum state

- Symplectic structure : $\Omega = \frac{1}{2} F_{\mu\nu} dk^{\mu} \wedge dk^{\nu}$
- Riemann structure : $G = g_{\mu\nu} d\lambda^{\mu} d\lambda^{\nu}$
- Complex structure : $J = FG^{-1}$ so that $J^2 = -I_d$

$$\begin{pmatrix} 0 & \frac{2}{\sinh 2\beta} \\ -\frac{1}{2}\sinh 2\beta & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\sinh 2\beta \\ -\frac{1}{2}\sinh 2\beta & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{4}{\sinh^2 2\beta} \end{pmatrix}$$

QPT → complex partially negative (pseudo-Riemann)

Pseudo-Kähler structure

Compatibility is still preserved.

Conclusion & Conjecture

- Complex eigenmodes (frame) across the entire parameter space.
- We defined the QGT by introducing the covariant derivative.

Quantum geometric structure of QPT system with SU(1,1) Hamiltonian

Kahler structure		bosonic QPT	*Pseudo-Kahler structure*
I) Complex structure ……	J		Complex structure
II) symplectic structure …	Ω		Symplectic structure
III) Riemann metric …	G		Pseudo Riemann metric

Pseudo-Kähler structure …. Related to geometry of dissipative systems, vacuum instability, quantum gravity, and so on!

Vacuum parametric amplification by a chirped pulse with CEP control

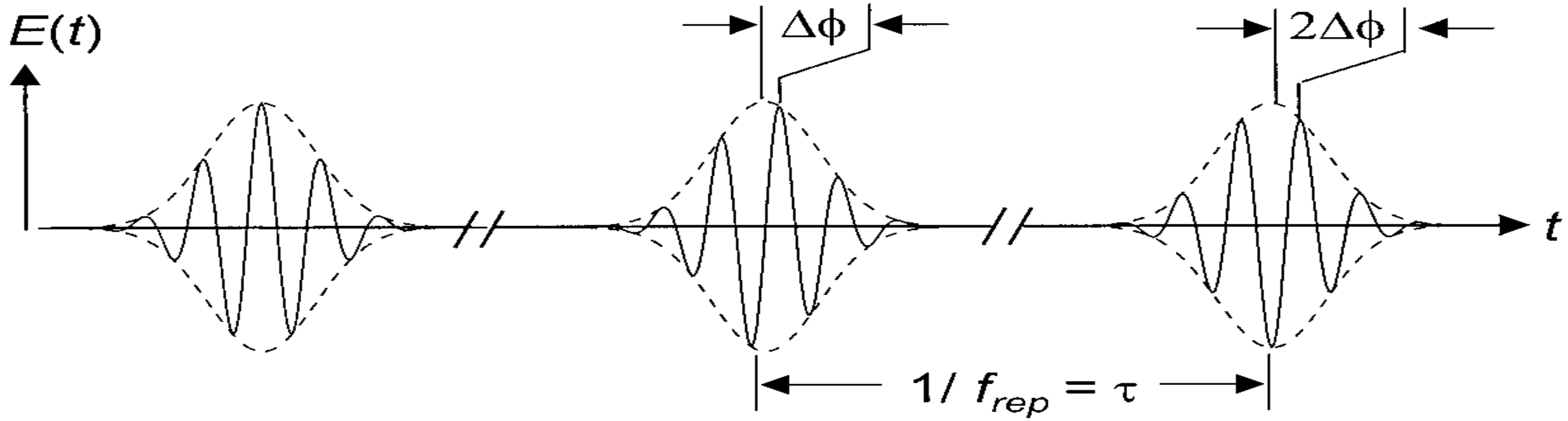
$$\hat{H} = \frac{\omega}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) + \frac{f}{2}\left(e^{i\theta}\hat{a}^2 + e^{-i\theta}\hat{a}^{\dagger 2}\right)$$

detuning frequency : $\omega \Leftrightarrow k_3 = \omega_0 - 2\omega_{\rm ex}$

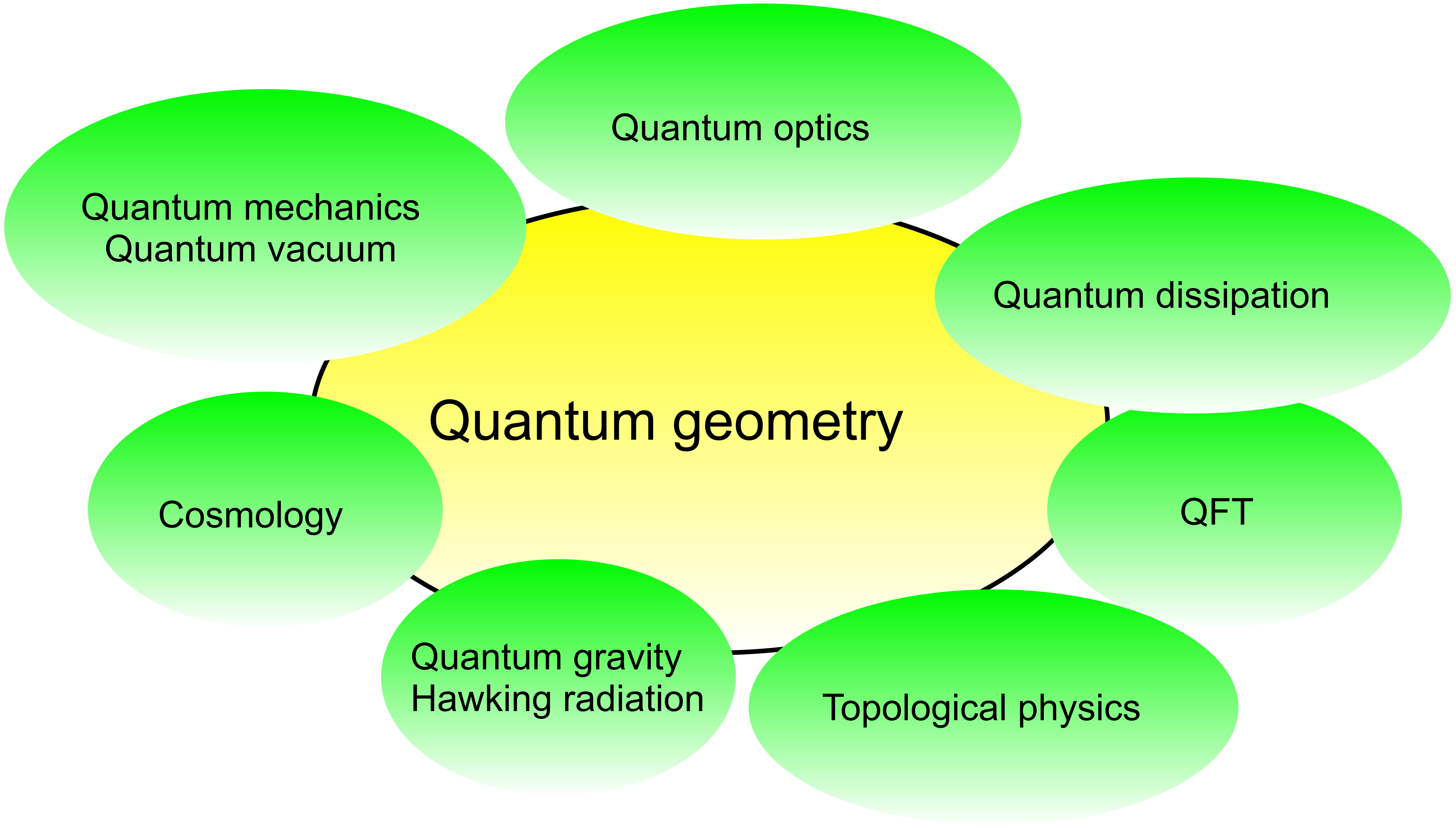
Amplitude of the external field : $f \Leftrightarrow k_\parallel$

Carrier-envelope-phase control : θ

pulsed chirped pulse with CEP control



Quantum geometry is a universal language that bridges various fields.



Thank you for your attention