Periodic Boundary Condition of Partons in the Kitaev Model

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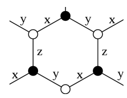
The Kitaev honeycomb model is an exactly solvable model that hosts a quantum spin liquid ground state, a novel phase of matter characterized by non-trivial many-body entanglement. This phase exhibits phenomena such as excitations with fractional statistics and topological features. Parton construction can decouple the up and down spin sectors of the Kitaev honeycomb model, leading to the convenience of theoretical analysis. Many literatures, including Kitaev's original work, analyze the model with infinite lattice sites, but investigating finite-size systems is necessary for both numerical studies and computing physical quantities for experiments. However, when doing so, some issues related to the periodic boundary condition arise due to the emergent gauge redundancy. In this research, we investigate the periodic boundary condition of partons in the Kitaev honeycomb model. The result turns out that the parton solution highly depends on the type of periodic boundary condition we chose. This research may help us to understand more about the properties of the parton technique, especially its application to the analysis of quantum spin liquids.

Introduction

- Kitaev honeycomb model [1]
- $\sigma_i^\mu \to i b_i^\mu c_i, \quad \hat{u}_{ij} \equiv i b_i^{\alpha_{ij}} b_j^{\alpha_{ij}}$ ($\mathbb{Z}_2$ gauge field $\hat{u}_{ij}^2 = 1$)

$$\hat{H} = -\sum_{\langle ij \rangle} J_{\alpha_{ij}} \sigma_i^{\alpha_{ij}} \sigma_j^{\alpha_{ij}} \to i \sum_{\langle ij \rangle} J_{\alpha_{ij}} \hat{u}_{ij} c_i c_j$$



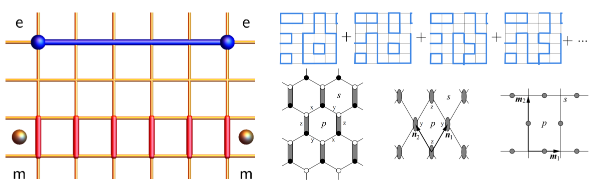
The eigenstate can be separated into two parts: $|\psi\rangle = |M\rangle |\{u_{ij}\}\rangle$ because of the facts that $[c_i, \hat{u}_{kl}] = 0$, $[\hat{u}_{ij}, \hat{u}_{kl}] = 0$

- Kitaev model in the **anisotropic limit** ($J_z \gg J_x, J_y$) is effectively equivalent to the toric code model [1]

$$\hat{H}_{\text{toric}} = -K \sum_p \hat{Q}_p - K \sum_s \hat{Q}_s$$

- Abrikosov fermion (**parton**) representation [2]: $\sigma_i^\mu = f_{\alpha i}^\dagger \tau_{\alpha\beta}^\mu f_{\beta i}$

$$f_{i\downarrow} = \frac{1}{2}\left(-b_i^y + i b_i^x\right) \quad , \quad f_{i\uparrow} = \frac{1}{2}\left(-c_i + i b_i^z\right)$$



- Properties of topological order for gapped systems: (i) ground state degeneracy, (ii) anyon, (iii) non-trivial entanglement.

Motivation

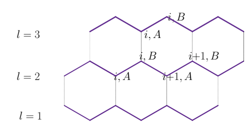
►It has been known that the **topological order (TO)** is a consequence of the nontrivial **entanglement structure (ES)** [3]. On the other hand, **parton construction** is one of the useful tool for studying quantum spin systems. However, relating TO and ES through parton construction is not obvious.

►**Finite size** analysis is inevitable while performing computational simulation. In this scenario, **periodic boundary condition (PBC)** and **antiperiodic boundary condition (APBC)** of partons become important [4].

►Before the general discussion, it would be easier to **start with the anisotropic limit** and investigate **how to use those parton techniques to get the same result as the toric code model**.

Structure of the Wave Function

$$\hat{H}_{MF} = i \sum_{\langle ij \rangle} J_{\alpha_{ij}} u_{ij} c_i c_j - |K| \sum_{\langle ij \rangle \in z} u_{ij} \hat{u}_{ij} - |K| \sum_{\langle ij \rangle \in x \cup y} u_{ij} \hat{u}_{ij}$$



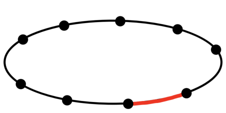
- Fully separated up and down sectors make it valid to express the wave function in the form of $|\psi\rangle = |\psi_\downarrow\rangle |\psi_\uparrow\rangle$
- Gutzwiller projection should be applied to obtain the physical states (no vacancy or double occupancy)

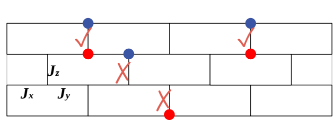
$$|\psi_{\text{phys}}\rangle = \mathcal{P}_G |\psi_\downarrow\rangle |\psi_\uparrow\rangle = \prod_i \left(n_{i\uparrow} + n_{i\downarrow}\right)\left(2 - n_{i\uparrow} - n_{i\downarrow}\right) |\psi_\downarrow\rangle |\psi_\uparrow\rangle$$

Result

$$f_{\sigma,\mu,i+N_l} = \begin{cases} f_{\sigma,\mu,i} & ; (PBC) \\ -f_{\sigma,\mu,i} & ; (APBC) \end{cases} \qquad \prod_{\langle ij \rangle \in l} \left(\frac{\mathbb{1} + u_{ij}\hat{u}_{ij}}{2}\right) = \frac{1}{2^{2N^{(l)}}} \mathcal{P}'_u \left[\mathbb{1} - (-1)^{\hat{N}_\downarrow^{(l)}} \prod_{\langle ij \rangle \in l} u_{ij}\right]$$

$$|\psi_\uparrow\rangle = \bigotimes_{\mathbf{x}} \frac{1}{\sqrt{2}} \left(|0\rangle_{\mathbf{x}A} |0\rangle_{\mathbf{x}B} + i |\uparrow\rangle_{\mathbf{x}A} |\uparrow\rangle_{\mathbf{x}B}\right) \quad \text{(anisotropic (toric code) limit solution)}$$





$$\begin{cases} \text{PBC} \begin{cases} \text{Even number of } \downarrow \text{ fermions in } A \\ \text{Odd number of } \downarrow \text{ fermions in } B \end{cases} \\ \text{APBC} \begin{cases} \text{Even number of } \downarrow \text{ fermions in } A \\ \text{Odd number of } \downarrow \text{ fermions in } B \end{cases} \end{cases}$$

- Gauge field configuration reflects PBC and APBC of partons, like the magnetic flux insertion into a ring.
- Total number of down-spin fermions in a layer $\hat{N}_\downarrow^{(l)}$ is determined by the type of periodic boundary condition $\prod_{\langle ij \rangle \in l} u_{ij} = \pm 1$ in the way that $(-1)^{\hat{N}_\downarrow^{(l)}} = -\prod_{\langle ij \rangle \in l} u_{ij}$.
- Under the isotropic (toric code) limit, the up-spin sector as well as the Gutzwiller projection impose a constraint that two spins in the same z-link have to be parallel ($|\uparrow\rangle|\uparrow\rangle$ or $|\downarrow\rangle|\downarrow\rangle$).
- **PBC (APBC)** and **number parity** of fermions in different sublattices give four **topological** ground states (can't be mixed).

Further Questions

1. A relaxed toric code limit (still gapped but soften)? In this scenario, there would be additional fluctuations in up-spin wave function, but the topological property is supposed to be unchanged. How to describe it?
2. How to generalize the picture to other spin systems? Is this work a special case only for the Kitaev model and the toric code model?

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References

[1] A. Kitaev, *Ann. Phys.* **321**, 2 (2006)

[2] F. J. Burnell and C. Nayak, *Phys.Rev.B* **84**, 125125 (2011)

[3] X. G. Wen, *Phys. Lett. A* **300**, 175 (2002)

[4] S. P. Kou, M. Levin, X. G. Wen, *Phys. Rev. B* **78**, 155134 (2008)

Conclusions

►In this work, we have shown that the **periodic boundary condition of partons** is crucial in a **finite size system**. One should also notice that the ground state degeneracy can only exist in a finite system.

►Neither $|\psi_\uparrow\rangle$ nor $|\psi_\downarrow\rangle$ possess the topological order. It is the **periodic boundary condition** as well as the **Gutzwiller projection** give that pattern. At this point, one of the relations between the **parton construction** and the **topological order**, the **PBC (APBC)**, is pointed out.