Gravitational Dressing Operators and Radiative Observables from Binary Systems

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Gravitational waves : Status & Challenges

Detections thus far

- Over the past decade, the *Laser Interferometer Gravitational-wave Observatory* (LIGO) collaboration have made about 200 detections from binary gravitating systems involving black holes and neutron stars.
- Ground based detectors: GWs with strain (magnitude) from 10^{-20} to 10^{-22} and frequencies in the hertz range.
- This corresponds to events involving 10 to 100 solar mass black holes
- Current (O4) run operates with the LIGO Hanford and Livingston detectors (US), the Virgo detector (Italy) and KAGRA (Japan)

LIGO Lab/Virgo

Forthcoming detections

Laser Interoferometer Space Antenna (LISA) (2030's)

- 10^{-16} to 10^{-21} amplitude in 10^{-3} Hz range
- Extreme mass ratios, unbounded events

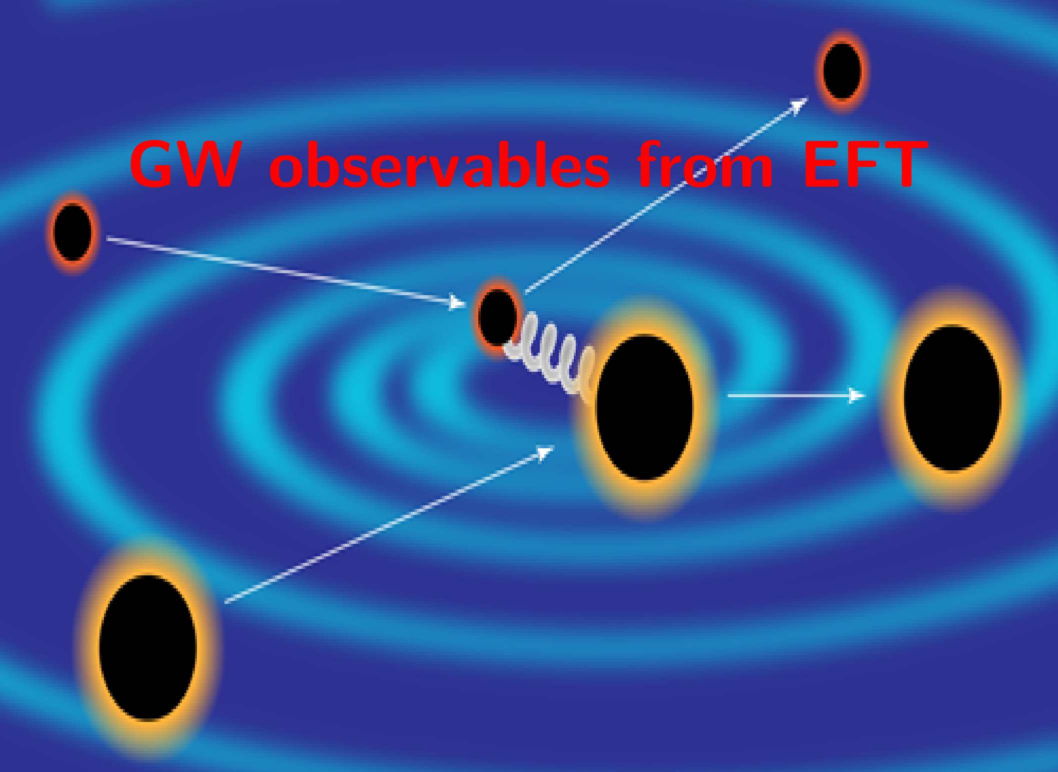
Pulsar Timing Arrays (PTA) (2020 -)

- 10^{-14} to 10^{-16} amplitude in 10^{-9} Hz range
- Super massive black holes, stochastic GW background

Future/planned detections –

- Unbound binary systems – a scattering problem
- *Extreme mass ratios* with $\frac{M_2}{M_1} << 1$.
- Gravitational waves from spacetime background

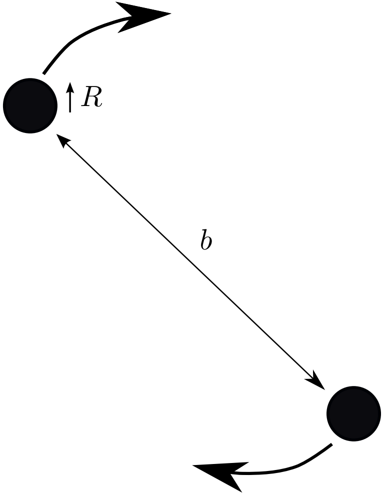
GW observables from EFT



The binary scattering/inspiral cannot be solved exactly.

We require perturbative solutions (expansion in small parameter)

There are two broad cases:



1. Small G expansions where

$$\begin{array}{ccc} \text{Distance} & >> & \text{Size} \\ b & >> & R \end{array}$$

$$\Rightarrow \frac{GM_{1,2}}{c^2 b} << 1$$

2. Self-force expansion in

$$\lambda = \frac{M_1}{M_2} << 1$$

Valid to all orders in G

Small G approximations

- Post-Newtonian (PN) approximation, which assumes

$$\frac{GM}{c^2 b} \approx \frac{v^2}{c^2} << 1$$

  Follows from virial theorem (Kinetic energy $=$ Potential energy);
  Applicable in the early inspiral for the bound case

- PN: small $v\,, G$; expansion up to $v^{2n} \Rightarrow$ n PN order

- Post-Minkowski (PM) approximation $\dfrac{GM}{c^2 b} << 1.$

- PM: small G; expansion to $G^n \Rightarrow$ n PM order.
  Applicable to bound/unbound cases, for all velocities

- In overlap of validity, PM and PN have consistent expansions.
  Self-force results part of answer (0SF at 3PM and 4PM; 1SF at 5PM and 6PM, etc.)

	0PN	1PN	2PN	3PN	4PN	5PN	6PN	7PN		
1PM	$(1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ v^{12}$	$+\ v^{14}$	$+\ \dots)$	G^1
2PM		$(1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ v^{12}$	$+\ \dots)$	G^2
3PM			$(1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ v^{10}$	$+\ \dots)$	G^3
4PM				$(1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ v^8$	$+\ \dots)$	G^4
5PM					$(1$	$+\ v^2$	$+\ v^4$	$+\ v^6$	$+\ \dots)$	G^5
6PM						$(1$	$+\ v^2$	$+\ v^4$	$+\ \dots)$	G^6

$$\vdots$$

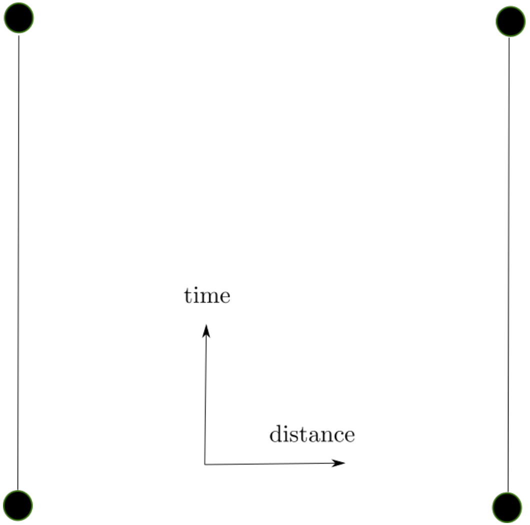
Pre-2019 developments

- Classical PM solutions of Einstein's equations among earliest attempted. 2PM solutions derived till 1980's. Problem was not tractable to higher orders due to ill defined integrals
- PN solutions were easier to derive, and favorable for creating bound state waveform templates. Results were derived till 4PN (spinless) and 2.5 PN (spinning)

  [Currently up to 6PN (spinless), 3PN (spin)]
- Key observables of interest: Two body potential, momentum and angular momentum change of objects, radiative observables.
- Potential has 'conservative' and 'dissipative' parts
- *Conservative*: Time symmetric part (under $t \to -t$).
- *Dissipative*: Time asymmetric part, due to radiation.
- Recent breakthroughs in higher PM orders from an unlikely place – scattering amplitudes!

Binary black hole scattering: Widely separated

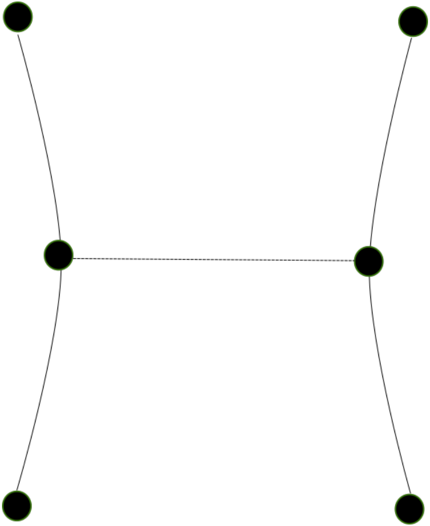
To see why, let's consider two black holes moving from past to future



If very widely separated, they don't interact

Binary black hole scattering: Leading interaction

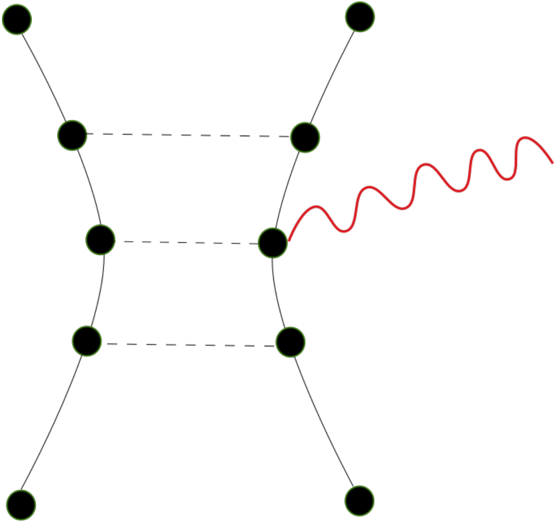
As they get closer, they interact with small corrections to Newton's law



The interaction through the gravitational potential is indicated through an instantaneous exchange. The trajectory gets corrected from straight lines

Binary black hole scattering: Subleading interactions

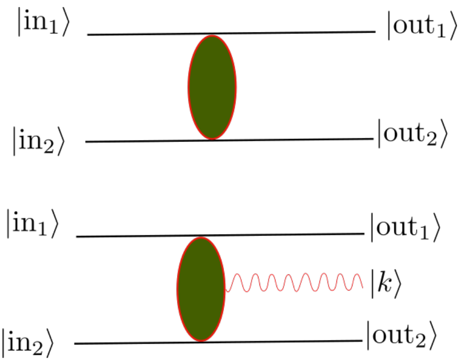
Closer still, there are conservative interactions from the gravitational potential, as well as gravitational wave radiation from dissipation



The red wave is the emitted radiation

Scattering amplitudes

This is beginning to look a lot like scattering amplitudes



$\hat{S} = 1 + i\hat{T}$ is the S-matrix,
$\hat{T}$ gives the scattering amplitude

$$|\text{out}\rangle = \hat{S}|\text{in}\rangle$$

Expectation values of a radiative observable $\mathcal{O}$

$$\langle\text{out}|\mathcal{O}|\text{out}\rangle = \langle\text{in}|\hat{S}^\dagger\mathcal{O}\hat{S}|\text{in}\rangle$$

will produce an outgoing radiation state $|k\rangle$

Lastly, the blob are possible loop contributions. A n-loop exchange gives a G^{n+1} correction to the scattering (0-loop is also called tree level).

Thus we can get PM results. The major issue : All of this is quantum, and we want classical physics!

Effective Field Theory for gravitational waves

The thing that saves the program are relevant length scales in the problem, that allow us to extract classical observables

- The black holes/compact objects are widely separated $(b >> R)$
- Well approximated by point particles, with additional corrections.
- The other consequence is a large angular momentum for the system

$$J \sim b \times p >> 1$$

- As a result, we have the *correspondence principle* for incoming and outgoing *states* ($|\text{in}_{1,2}\rangle$ and $|\text{out}_{1,2}\rangle$), with a (nearly) continuous spectrum from large quantum numbers.
- Classical external states also require no new particle production.
- For objects with radii larger than their de Broglie wavelength

$$R >> \lambda = \frac{\hbar}{p},$$

we won't have particle production effects from the vacuum

Effective Field Theory for gravitational waves

- There are gravitons exchanged in the scattering process. Due to large b, its conjugate momentum q is small.
- Two possible types of exchanged gravitons with these length scales – those that are instantaneous (called 'potential' gravitons) , and truly radiative gravitons
- The emitted gravitational waves have a long wavelength for the same reason (large b)
- We thus have a scattering with external states that have a large energy (squared), let's call it s. The exchanged momentum q is small. The limit

  $$\frac{s}{q^2} << 1$$

  is called the *eikonal approximation*, and gives us eikonal amplitudes from a general scattering amplitude.

Eikonal amplitude details [Review arXiv:2306.16488 [hep-th]]

- Call the incoming state $|p_1\,, p_2\rangle = |p\rangle$ and $|p_3\,, p_4\rangle = |p'\rangle$. From $\hat{S} = 1 + i\hat{T}$, we then find

$$\langle p'|S|p\rangle \sim 1 + 2\pi i\delta(|\vec{p}| - |\vec{p}'|)\mathcal{A}$$

$$\mathcal{A} = e^{2i\delta}(1 + 2i\Delta) \quad \text{when} \quad \frac{s}{q^2} << 1$$

  with δ the eikonal phase and Δ a quantum remainder.
- $\tilde{A}$, δ and Δ expand in powers of G

$$i\tilde{A}_0 = 2i\delta_0\,, \quad i\tilde{A}_1 = 2i\delta_1 + \frac{1}{2}(2i\delta_0)^2 + 2i\Delta_1 + \cdots$$

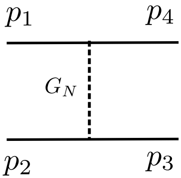
$$i\tilde{A}_2 = (2i\delta_2 + 2i\delta_0 2i\Delta_i) + \frac{1}{3!}(2i\delta_0)^3 + (2i\delta_0)(2i\delta_1) + \cdots$$

- The suffix indicates orders of G_N in the exchange

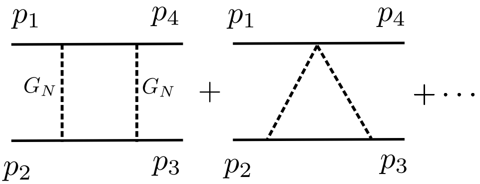
$$\delta_0 \sim G\;; \quad \delta_1 \sim G^2\;; \quad \delta_2 \sim G^3$$

- $\tilde{A}_0$, $\tilde{A}_1$ and $\tilde{A}_2$ give the tree level (G or 1PM), 1-loop (G^2 or 2PM) and 2-loop (G^3 or 3PM) results respectively, with contributions from...

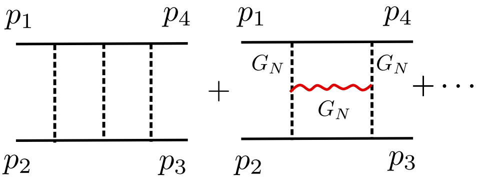
- Tree level



- 1-loop



- 2-loop



δ_2 properties

- The 2-loop amplitude has radiative exchanges between loops
- This gets realized as an imaginary contribution in the eikonal phase – a coherent gravitational dressing
- A consequence of inelastic exchanges – energy, momentum and angular momentum is lost in the process as gravitational waves.
- In the very low frequency limit, we have the result

$$e^{i\mathrm{Im}2\delta_2}\, e^{i\mathrm{Re}2\delta_2} \xrightarrow[\omega\to 0]{} \exp[-\Delta_1^\kappa]\, e^{i\mathrm{Re}2\delta_2}$$

- $\exp[-\Delta_1^\kappa]$ is the Weinberg soft graviton dressing

$$\Delta_1^\kappa = \frac{1}{\hbar}\int_{\vec{k}} d^3k \left(a_i(k) f_i^*(k) - a_i^\dagger(k) f_i(k)\right)$$

$$f_i(k) = \varepsilon^*_{i\,,\mu\nu}(k) F^{\mu\nu}(k)\ ; \qquad F^{\mu\nu}(k) = \sum_n \frac{\kappa p_n^\mu p_n^\nu}{k\cdot p_n - i0_k}$$

Gravitational wave observables

- Expectation values of graviton mode operators $\mathcal{O}$ with respect to the vacuum will give gravitational wave observables.
- If we consider the low frequency limit of the eikonal operator, the corresponding observable is the leading late time result (since frequency and time are inversely related)

$$\langle 0|e^{-i\mathrm{Re}2\delta_2}e^{\Delta_1^\kappa}\mathcal{O}e^{-\Delta_1^\kappa}e^{i\mathrm{Re}2\delta_2}|0\rangle = \langle 0|e^{\Delta_1^\kappa}\mathcal{O}e^{-\Delta_1^\kappa}|0\rangle := \langle \mathcal{O}\rangle_{\Delta_1^\kappa}$$

- We can consider the mode and momentum operators for the graviton to derive the waveform and emitted momentum

$$h_{\mu\nu}(x) = \int_{\vec{k}} d^3k \left[a_i(k)\varepsilon_{i\,,\mu\nu}(k)e^{ik.x} + a_i^\dagger(k)\varepsilon^*_{i\,,\mu\nu}(k)e^{-ik.x}\right]$$

- Waveform contribution : $2\kappa\langle h_{\mu\nu}\rangle_{\Delta_1^\kappa} = W^G_{\mu\nu}$

$$P^\alpha(x) = \int_{\vec{k}} d^3k\, k^\alpha a_i^\dagger(k)a_i(k)$$

- $\langle P^\alpha\rangle_{\Delta_1^\kappa} = \mathcal{P}^{\alpha\,;G}$ and $\dfrac{dE^G}{d\omega} = \dfrac{d\mathcal{P}^{0\,;G}}{d\omega}$

δ_3 and higher issues

- The generalization of the coherent state is not known beyond 2 loops
- The 2PM waveform has been derived from scattering amplitude methods
- However , other radiative observables, and their derivation becomes increasingly complicated to higher PM (and thus higher loop) orders
- There fortunately is a way to infer the low frequency dressing for eikonal amplitudes from another related approach – the worldline formalism
- This provides a roadmap to infer late time observables to higher PM orders, and possibly generalize beyond the standard PM case to self-force problems

Gravitational Dressing & Radiative Observables

Following [KF, F-L. Lin JHEP **06** (2024) 15]

$$S = -\int d^4x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + m^2\phi^*\phi\right)$$

- Consider $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$ with $\kappa^2 = 8\pi G$ and $p_\mu = -i\partial_\mu$

  to get $\qquad S := -\int d^4x\phi^*\left(2\hat{H}\right)\phi$

- $\hat{H}$ has a 'free' part and interactions to all orders in κ

$$2\hat{H}(x\,,p) = p^2 + m^2 + 2\kappa\hat{H}^\kappa(x\,,p) + 4\kappa^2\hat{H}^{\kappa^2}(x\,,p) + \mathcal{O}(\kappa^3)$$

- Worldline formalism (Schwinger): Identify $\left(\hat{H}(x\,,p) - i\epsilon\right)^{-1}$ with scalar field propagator (in background gravitational field)

- Basically follows from using

$$\frac{1}{H} = \int_0^\infty dte^{-Ht}\,, \qquad \langle p|x\rangle \sim e^{-\frac{i}{\hbar}p.x}\,,$$

  and basic path integral representation methods (introduce an discrete set of complete basis states, and take the continuum limit)

- This gives the external particle 'propagator' from an initial position to final momentum

$$\left\langle p_f \right| \left(\hat{H} - i\epsilon\right)^{-1} \left| x_i \right\rangle = \int_0^\infty dT \int_{x(0)=x_i}^{p(T)=p_f} \mathcal{D}p\mathcal{D}x$$
$$\exp\left[-\frac{i}{\hbar}p(T).x(T) + \frac{i}{\hbar}\int_0^T dt\left(p\dot{x} - \hat{H}(x\,,p) + i\epsilon\right)\right],$$

- This can't be solved exactly, but we can do so about known solutions

- We consider

$$x(t) \to x_i + p_f t + x(t)\,, \qquad p(t) \to p_f + p(t)\,, \qquad x(0) = 0 = p(T)$$

- These correspond to fluctuations about eikonal trajectories, and will provide a soft graviton dressing

- The propagator can be expressed as

$$\left\langle p_f \right| \left(\hat{H} - i\epsilon\right)^{-1} \left| x_i \right\rangle = \int_0^\infty dT e^{-i(p_f x_i + (p_f^2 + m^2 - i\epsilon)T)} f(T)$$

- Consider $x_i = 0$. With $x_i \neq 0$ we get a orbital angular momentum contribution. Relevant for subleading soft graviton factors at order κ and their generalizations to $\mathcal{O}(\kappa^2)$
- Amputated propagator gives the dressing

$$\lim_{p_f^2 \to -m^2} -i(p_f^2 + m^2 - i\epsilon)\left\langle p_f \right| \left(\hat{H} - i\epsilon\right)^{-1} \left| x_i \right\rangle = \lim_{T \to \infty} f(T)$$

- $f(T)$ involves the double path integral over $p(t)$ and $x(t)$. The late time limit gives the soft graviton dressing that we are after.
- We will consider the four external particle case as before $(f(T) \to \prod_{i=1}^4 f_i(T))$, and we consider soft graviton modes in terms of the their creation and annihilation operators $(h_{\mu\nu} \to \hat{h}_{\mu\nu})$.

- The term from H^{κ^2} now gives a *double soft graviton* contribution.
- Subtlety: This dressing is manifestly invariant when the two gravitons are collinear (which we assume).
- This means that if the two gravitons have four-momenta k and l respectively, we have

$$k^2 = 0 = l^2 \,, \qquad k.l = 0$$

- We arrive at

$$\prod_{i=1}^{4} f_i(\infty) = \exp[-\Delta] = \exp[-\Delta_1^{\kappa} - \Delta_2^{\kappa^2}]$$

- The Δ_1^{κ} is in terms of the Weinberg soft factor (as before), while

$$\Delta_2^{\kappa^2} = \frac{1}{2\hbar}\int_{\vec{k}} d^3k \int_{\vec{l}} d^3l \left[a_i^{\dagger}(k)a_j^{\dagger}(l)A_{ij}(k,l) - a_i(k)a_j(l)A_{ij}^{*}(k,l) \right.$$
$$\left. + a_i^{\dagger}(k)a_j(l)B_{ij}^{*}(k,l) - a_j^{\dagger}(l)a_i(k)B_{ij}(k,l)\right](2\pi)^2\delta(\Omega_k\,,\Omega_l)$$

Factorization

- Using the Baker-Campbell-Haussdorf formula, we find

$$\exp[-\Delta] = \exp[-\Delta_1]\exp[-\Delta_2^{\kappa^2}]\,, \qquad \Delta_1 = \sum_{n=0}^{\infty}\Delta_1^{\kappa^{2n+1}}$$

- This identifies a subleading double graviton correction for Δ_1^{κ} in the coherent dressing

$$\Delta_1^{\kappa^3} = \frac{1}{2\hbar}\int_{\vec{k}} d^3k \int_{\vec{l}} d^3l \left[a_i^\dagger(k)\left(A_{ij}(k,l)f_j^*(l) + B_{ij}^*(k,l)f_j(l)\right)\right.$$
$$\left. - a_i(k)\left(A_{ij}^*(k,l)f_j(l) + B_{ij}(k,l)f_j^*(l)\right)\right](2\pi)^2\delta(\Omega_k\,,\Omega_l)$$

- We have a gravitational dressing derivation from the wordline, with assumptions that are consistent with those for eikonal amplitudes .
- We thus infer that the 3 loop low frequency approximation for the imaginary part of the eikonal phase takes the form

$$e^{i2\delta_3} \xrightarrow[\omega\approx 0]{} \exp[-\Delta]e^{i\mathrm{Re}2\delta_3}$$

Expectation values

- This suffices to determine expectation values for purely gravitational observables $\mathcal{O}$

$$\langle 0| e^{\Delta_1} e^{\Delta_2^{\kappa^2}} \mathcal{O} e^{-\Delta_2^{\kappa^2}} e^{-\Delta_1} |0\rangle := \langle \mathcal{O} \rangle_\Delta$$

- Expectation values wrt $e^{\Delta_2^{\kappa^2}}$ vanish classically.

- Non-vanishing results follow from canonical commutation relations. However, since $[a\,, a^\dagger] \sim \hbar$, this is subleading in the $\hbar \to 0$ limit.

- Hence the coherent part of the dressing contributes

$$\langle \mathcal{O} \rangle_\Delta = \langle \mathcal{O} \rangle_{\Delta_1^\kappa} + \langle \mathcal{O} \rangle_{\Delta_1^{\kappa^3}}$$

- $\langle \mathcal{O} \rangle_{\Delta_1^{\kappa^3}}$ provide 2PM next-to-eikonal corrections to the known 1PM Weinberg dressing results

2PM and higher PM observables

Proceeding as before for expectation values, we find

- Waveform contribution : $2\kappa\langle h_{\mu\nu}\rangle_\Delta = W^{G}_{\mu\nu} + W^{G^2}_{\mu\nu}$
- Emitted momentum: $\langle P^\alpha\rangle_\Delta = \mathcal{P}^{\alpha\,;G} + \mathcal{P}^{\alpha\,;G^2}$
- $\langle L^{\alpha\beta}\rangle_\Delta = \mathcal{L}^{\alpha\beta\,;G} + \mathcal{L}^{\alpha\beta\,;G^2}\ ; \quad \langle S^{\alpha\beta}\rangle_\Delta = \mathcal{S}^{\alpha\beta\,;G} + \mathcal{S}^{\alpha\beta\,;G^2}$
- $W^{G}_{\mu\nu}$ had the result of a linear memory effect – the permanent, late time change in the spacetime after the scattering event.
- $W^{G^2}_{\mu\nu}$ gives the *non-linear* memory effect, also known as the Christodolou effect. Here, the initial emission of gravitons act a a source for spacetime changes before asymptotic times. Our result exactly matches this prediction, and provides the first derivation of this effect from scattering amplitude methods.

Summary

- Gravitational wave observables can be derived from the classical limit of scattering amplitudes.
- The effective field theory for binary black hole scattering involves eikonal amplitudes with a gravitational dressing operator. However their formal derivation beyond 2 loop order is not known.
- We used the worldline formalism to furnish this gravitational dressing, in a low frequency expansion. This was used to derive radiative observables, which inclue the waveform, emitted momentum and angular momentum, to 3 loop order.
- The resulting dressing has the form of a squeezed coherent state.
- One of the planned directions is to better understand properties of the squeezing operator in the context of black hole scattering

Thank You