Review on differential forms.

i) Basics on top. manifolds

top. Manifold $\mathcal{M}$ is a top. space s.t. $\forall$ $p \in M$ There is a neighborhood which is homeomorphic to a disc in $\mathbb{R}^n$.

Homeomorphism: Continuous bijection w/ inverse. (not necessarily structure) preserving.

$\Rightarrow$ i) Bijection (one-to-one & onto)

ii) continuous

iii) f^{-1} exists & is continuous.

"Being Homeomorphic" is simply saying that There is an equiv. Relation.

Under This definition: $\mathcal{M} = \bigcup_i U_i$; $\{U_i\}$: collection of open covers of $\mathcal{M}$.

The map.

$$\varphi_U : U \longrightarrow D^n$$

D^n → open disc

φ_U ↓ assigns local coordinates on $\mathcal{M}$.

The points in D^n (assigned by φ_U) are local Coordinates -

each such map φ_U is called a chart.

Further consider U, V as open sets in $\mathcal{M}$. with $U \cap V \neq \emptyset$.

we can have

$$\varphi_U : U \to D^n \quad \& \quad \varphi_V : V \to D^n.$$

on the intersection we have two choices of coordinates. They must be related somehow.
Consider the composition: $\underline{\varphi_U \varphi_V^{-1}}$ (Transition function)
The transition function tell us how to go from V-coordinates to U-coordinates.

⊛ A manifold is differentiable if all transition functions are C^∞.
↳ have well defined derivatives of arbitrarily high order.

C^∞ maps are called diffeomorphisms.

<u>example:</u> S^1 (a circle), $S^1 = M$.

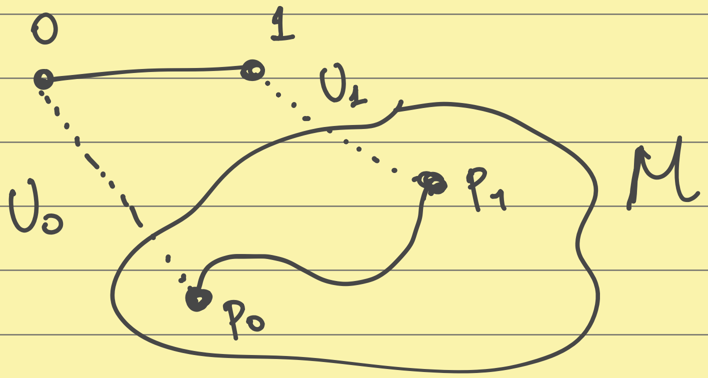
S^1 is labeled by a coordinate θ. which obeys
$\theta \sim \theta + 2\pi \Rightarrow 0, 2\pi$ are the same point!

$\Rightarrow S^1$ cannot be covered by a single chart
otherwise we cannot define good one-to-one maps!

Vector Fields on $\mathcal{M}$.

Curves on $\mathcal{M}$: $\quad U_t : t \in [0,1] \longrightarrow \mathcal{M}$

take an interval in $\mathbb{R}$ — send to a region in $\mathcal{M}$.



let $\mathcal{F}(\mathcal{M})$ denote the set of all differentiable functions in $\mathcal{M}$

Thus, if $f \in \mathcal{F}(\mathcal{M})$; $f : \mathcal{M} \longrightarrow \mathbb{R}^1$
and f is C^∞

we can Define:

$$(X \circ f)(p_0) \equiv \frac{d}{dt} f(U_t(p_0)) \Big|_{t=0}$$

X is the tangent vector to the curve $U_t(p)$.
if one considers all curves going through p.
we get a vector space at p.

$X \in T_p\mathcal{M}$ — tangent space to $\mathcal{M}$ at the point p.

if we consider the union of all tangent spaces i.e., $\bigcup_p T_p\mathcal{M}$ we get $T\mathcal{M}$ which is commonly called the <u>tangent Bundle</u>.

This is too Abstract. Let us look at coord.

in local coord. we have

$$\varphi(U_t(p)) = \underbrace{x^i}_{\text{coord. at point } p.} + t\underbrace{\xi^i(x)}_{\text{some directional vector that gives the direction of the curve}} + \dots$$

So, from the definition:

$$X \cdot f(x) = \frac{d}{dt} f(x_i + t\xi^i(x))$$

$$= \xi^i \frac{\partial f}{\partial x_i}$$

$$\Rightarrow X = \underbrace{\xi^i(x)}_{\text{vector components}} \underbrace{\frac{\partial}{\partial x^i}}_{\text{vector basis}} \quad \text{in these local coord.}$$

<u>Differential forms on M.</u>

<u>1-form.</u> $T_p^* M$ dual tangent space (cotangent space).

$dx^i \in T_p^* M$ (local basis of $T_p^* M$)

as usual:

$$\underbrace{\left(\frac{\partial}{\partial x^i}, dx^j \right)}_{\text{interior contraction.}} = \delta_i^{\ j} \quad \left(\text{This is a Definition}\right)$$

elements on $\cup_p T_p^* M$ are called diff. forms.

a 1-form. can be generically written as:

$$\omega = \omega_i(x)dx^i$$

$\downarrow$ differentiable functions.

The interior contraction of a vector field $X = \xi^i \frac{\partial}{\partial x^i}$ with a 1-form is:

$$i_X \omega = \left(\xi^i \frac{\partial}{\partial x^i}, \omega_j dx^j \right)$$

$$= \xi^i \omega_j \delta_i^{\,j} = \xi^i \omega_i$$

components of 1-forms are also called "covariant vectors".

⊛ exercise: show that a 1-form is a coordinate invariant quantity.

Differential k-forms.

Consider the k-fold tensor product.

$$(T_pM)^k = T_pM \otimes T_pM \otimes \cdots \otimes T_pM$$

an element in this space will be a rank k tensor at the point p.
& similarly for T_p^*M.
we will focus with a special class of objects that are elements of

$\Lambda_k(T_p^* M) \rightsquigarrow$ anti symmetrized product of cotangent spaces.

for $\omega \in \Lambda_k(T_p^* M)$, we have:

$$\omega = \frac{1}{k!} \underbrace{\omega_{i_1 i_2 \cdots i_k}}_{\text{anti symmetric Tensor.}} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

$\wedge$: The wedge product, exterior product, outer prod.

e.g.,
$$dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$$

$$\begin{aligned} dx^\mu \wedge dx^\nu \wedge dx^\rho = & \, dx^\mu \otimes dx^\nu \otimes dx^\rho + dx^\rho \otimes dx^\mu \otimes dx^\nu \\ & + dx^\nu \otimes dx^\rho \otimes dx^\mu \\ & - dx^\mu \otimes dx^\rho \otimes dx^\nu \\ & - dx^\nu \otimes dx^\mu \otimes dx^\rho \\ & - dx^\rho \otimes dx^\nu \otimes dx^\mu . \end{aligned}$$

The exterior product of two forms

$$\alpha = \alpha_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \, 1/p!$$
$$\beta = \beta_{j_1 \cdots j_k} dx^{j_1} \wedge \cdots \wedge dx^{j_k} \, 1/k!$$

is $\underbrace{\alpha \wedge \beta}_{(k+p)\text{-form.}} = \frac{1}{p!\,k!} \alpha_{[i_1 \cdots i_p} \beta_{j_1 \cdots j_k]} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}$

Q. what is the highest rank of a n-form on m-dimensional manifold?

Pullback of a k-Form

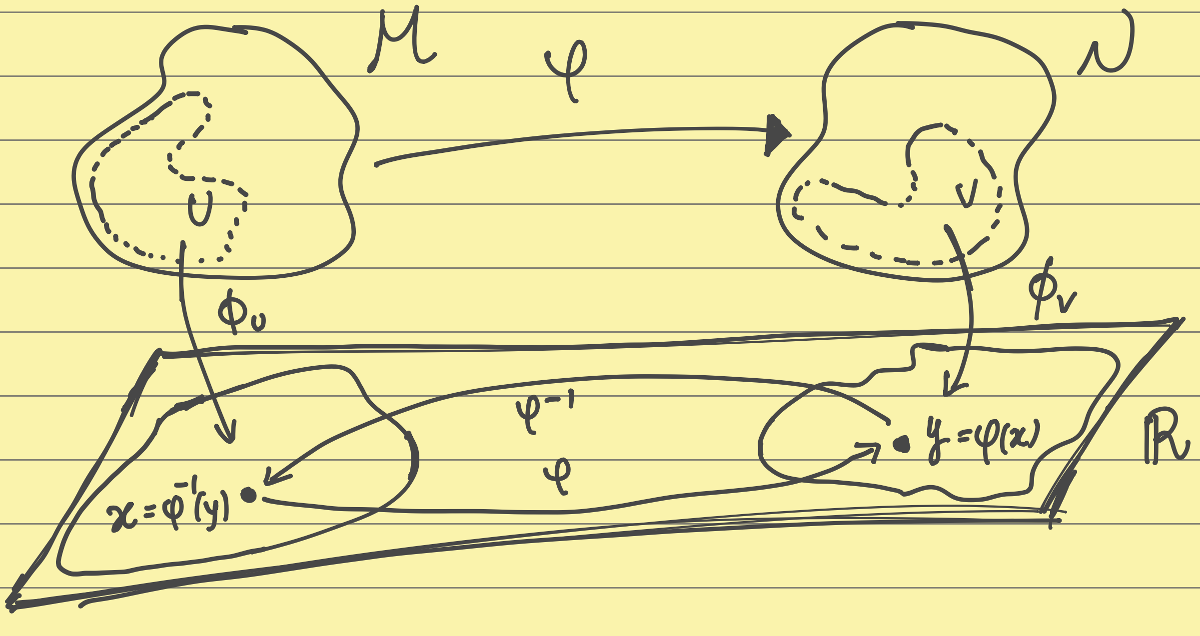
Consider a map from a manifold $\mathcal{M}$ to $\mathcal{N}$
$\varphi : \mathcal{M} \longmapsto \mathcal{N}$.

x: coordinates on $\mathcal{M}$; $\varphi(x)$ are coordinates of the image of $\mathcal{M}$ under φ.

ω : a k-form on $\mathcal{N}$ with local coordinates y.

$$\omega = \frac{1}{k!} \omega_{i_1 \cdots i_k}(y)\, dy^{i_1} \wedge dy^{i_2} \wedge \cdots \wedge dy^{i_k}$$

Goal: to find the corresponding k-form on $\mathcal{N}$.



The k-form on $\mathcal{N}$ with local coordinates y. Can be seen as a k-form on $\mathcal{M}$ as:

$$\varphi^* \omega = \frac{1}{k!} \omega_{i_1 \cdots i_k}(\varphi(x))\, d\varphi(x)^{i_1} \wedge \cdots \wedge d\varphi(x)^{i_k}$$

$$= \frac{1}{k!} \omega_{i_1 \cdots i_k}(\varphi(x)) \frac{\partial \varphi(x)^{i_1}}{\partial x^{j_1}} \frac{\partial \varphi(x)^{i_2}}{\partial x^{j_2}} \cdots \frac{\partial \varphi(x)^{i_k}}{\partial x^{j_k}}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

$$= \frac{1}{k!} \tilde{\omega}_{j_1 \dots j_k}(x)\, dx^{j_1} \wedge \dots \wedge dx^{j_k}$$

The form $\varphi^* \omega$ is therefore seen as a form on $\mathcal{M}$ (with coordinates x).

$\varphi^* \omega$ is referred as the pullback of ω on $\mathcal{N}$ by the map φ.

Similarly, if φ^{-1} exists, we can define a pushforward map that goes on the other direction.

<u>exterior derivative</u>

$$d : \Lambda_k(T^*\mathcal{M}) \longmapsto \Lambda_{k+1}(T^*\mathcal{M})$$

(maps k-forms to $k+1$-forms)

$$d\omega := \frac{1}{k!} \frac{\partial \omega_{i_1 i_2 \dots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$= \frac{1}{(k+1)!} \tilde{\omega}_{i_1 i_2 \dots i_{k+1}} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_{k+1}}$$

$$\tilde{\omega}_{i_1 i_2 \dots i_{k+1}} = \frac{\partial}{\partial x_1} \omega_{i_2 i_3 \dots i_{k+1}} - \frac{\partial \omega_{i_1 i_2 \dots i_{k+1}}}{\partial x_{i_2}}$$

$$+ \dots + (-1)^k \frac{\partial}{\partial x^{i_{k+1}}} \omega_{i_1 i_2 \dots i_{k+1}}$$

e.g., (i) Consider a 1-form $f = f_i dx^i$
take the exterior derivative

$$df = \frac{\partial f_i}{\partial x^j} dx^j \wedge dx^i$$

$$= \frac{\partial f_1}{\partial x^2} dx^1 \wedge dx^2 + \frac{\partial f_2}{\partial x^1} dx^2 \wedge dx^1$$

$$= \left(\frac{\partial f_1}{\partial x^2} - \frac{\partial f_2}{\partial x^1} \right) dx^1 \wedge dx^2$$

d is a generalization of the curl $\nabla \times$ operator!

Properties. (i) $d(\alpha + \beta) = d\alpha + d\beta$

(ii) $d(\alpha^p \wedge \beta^q) = d\alpha^p \wedge \beta^q + (-1)^p \alpha^p \wedge d\beta^q$

(iii) $d^2\alpha = d(d\alpha) = 0 \quad \forall$ forms α.

exercise: check all these properties.

exercise: Show that for a form ω, $d\omega$ also remains coordinate independent.

i.e.,

$$d\omega = \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i = \frac{\partial \tilde{\omega}_i}{\partial y^k} dy^k \wedge dy^i$$

(take ω a 1-form for simplicity).

Integration.

Consider a n-form on a n-dim manifold.

$$dx^{i_1} \wedge \dots \wedge dx^{i_n} = \epsilon^{i_1 i_2 \cdots i_n} \underbrace{dx_1\, dx_2 \dots dx_n}_{\text{volume element.}}$$

Therefore, a generic:

$$\omega = \omega_{i_1 i_2 \cdots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

$$= \omega_{i_1 i_2 \cdots i_n} \epsilon^{i_1 i_2 \cdots i_n} dx_1 \cdots dx_n$$

$$= \rho(x)\, d^n x .$$

$$\Rightarrow \int_{\mathcal{M}} \omega = \int_{\mathrm{vol}(\mathcal{M}(x))} \rho(x)\, d^n x$$

volume of the Region with x-coord. system.

<u>exercise:</u> show that under a change of variables.

$$dx^{i_1} \wedge \cdots \wedge dx^{i_n} = J\, dy^{i_1} \wedge \cdots \wedge dy^{i_n}$$

$\downarrow$

Jacobian matrix.

<u>exact & closed forms.</u>

if $d\omega = 0$, Then ω is said to be Closed.

if $\omega = d\alpha$, Then ω is said to be exact.

note that, if ω is exact, Then it is also closed. i.e.,

$$d\omega = d(d\alpha) = 0 \quad \text{from } d^2 = 0.$$

The converse is not true! (not generally)

<u>Poincaré Lemma.</u>

Every closed form, is locally exact.

i.e., if $d\omega = 0$, Then $\omega = d\alpha$ for some α in a local Patch of the manifold.

e.g., take $\mathcal{M} = S^1$.

Consider the 1-form $d\theta$.,

$d\theta$ is obviously closed (why?)
But θ is not exact.
$\theta \sim \theta + 2\pi$. (not globally well defined)
it is not a single-valued.

on the other hand. The 1-form

$\sin\theta d\theta$ is closed (Same reason as Before)
& is also exact:

$$\sin\theta d\theta = d(-\cos\theta);$$

since $\cos(\theta)$ is well defined on S^1.

stopped Here.

Stokes theorem.

if ω is a k-form & C is a $(k+1)$-dim subspace of $\mathcal{M}$.

$$\int_C d\omega = \int_{\partial C} \omega \qquad \partial C \sim \text{Boundary of } C.$$

Non-exactness of a form can be seen from this (in some cases):
take $\omega = \sin\theta \, d\theta \wedge d\varphi$ on $\mathcal{M} = S^2$.

$$\int_{S^2} \omega = 4\pi \quad (\text{integral of a solid Angle})$$

ω is closed. $d\omega = \frac{\partial \sin\theta}{\partial\theta} d\theta \wedge d\theta \wedge d\varphi = 0$

& if $\omega = d\alpha$ (exactness), Then:

$$\int_{S^2} \omega = \int_{S^2} d\alpha \underset{\text{Stokes Thm}}{=\!=} \int_{\partial S^2} \alpha = 0$$

S^2 is a closed manifold. $\partial S^2 = \emptyset$.

<u>Volume form & Hodge dual.</u>

Consider a Riemannian manifold $\mathcal{M}$ (we have a metric)

$$ds^2 = g_{ij}\, dx^i dx^j.$$

The volume element is $\sqrt{\det g}\, d^n x$

naturally, we can write

$$V = \frac{1}{n!} \sqrt{\det(g)}\, \epsilon_{i_1 i_2 \cdots i_n} dx^{i_1} \wedge \cdots \wedge dx^{i_n}$$

The volume form.

we can associate a $(n-k)$-form to a k-form w on an n-dim manifold. as follows

<u>Define:</u> $$T_\omega = \frac{1}{k!} \omega^{i_1 \cdots i_k} \frac{\partial}{\partial x^{i_k}} \cdots \frac{\partial}{\partial x^{i_1}}$$

$$= \frac{1}{k!} g^{i_1 j_1} \cdots g^{i_k j_k} \omega_{j_1 \cdots j_k} \frac{\partial}{\partial x^{i_k}} \cdots \frac{\partial}{\partial x^{i_1}}$$

$$g^{ij} = (g_{ij})^{-1}$$

(The indices are reversed for convenience. Avoid some minus signs.)

The Hodge dual of ω denoted as $\star\omega$ is defined:

$$\star\omega := (T_\omega, V)$$ ⇝ means contraction of the Tensors. $\left(\frac{\partial}{\partial x^i}, dx^j\right) = \delta_i^j$

$$= \frac{1}{(n-k)!}\left[\frac{1}{k!}\sqrt{\det(g)}\,\epsilon_{i_1\cdots i_k i_{k+1}\cdots i_n} g^{j_1 i_1}\cdots g^{j_k i_k} \times \omega_{j_1\cdots j_k}\, dx^{i_{k+1}}\wedge\cdots\wedge dx^{i_n}\right]$$

<u>check:</u> $\star(\star\omega) = (-1)^{k(n-k)}\omega$ Euclidean signature

$\star(\star\omega) = -(-1)^{k(n-k)}\omega$ Minkowski signature.

Hodge dual is metric dependent!

The Hodge dual allow us to define an inner product between forms.

$$(\alpha|\beta) = \int_M \alpha\wedge\star\beta$$

$$= \frac{1}{k!}\int_M \sqrt{\det g}\,\alpha_{i_1\cdots i_k}\beta_{j_1\cdots j_k} g^{i_1 j_1}\cdots g^{i_k j_k}.$$

$$= (\beta, \alpha).$$

For flat spacetime: $g_{ij} = g^{ij} = \mathrm{diag}(1,1)$

$$\Rightarrow \det g = 1$$

$$\Rightarrow \star\omega = \frac{1}{(n-k)!}\frac{1}{k!}\epsilon_{i_1 i_2\cdots i_k i_{k+1}\cdots i_n}\omega^{i_1 i_2\cdots i_k} \times dx^{i_{k+1}}\wedge\cdots\wedge dx^{i_n}$$

The Hodge is essentially the contraction w/ the Levi-Civita symbol & the corresponding form.

also:

$$(\alpha, \beta) = \int_{\mathcal{M}} \frac{1}{k!} \underbrace{\alpha_{i_1 i_2 \cdots i_n} \beta^{i_1 i_2 \cdots i_n}}_{\text{just the contraction of the form components.}} = (\beta, \alpha)$$

End of Review.