Minkowski Spacetime

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（Ⅰ）Main theory

Minkowski geometry(1/2)

■the spatial part of line element in Euclidean spacetime and Minkowski Spacetime are same, but the temporal part ct of line element in Euclidean spacetime and Minkowski Spacetime are different

$$\text{Euclidean:} ds^2 = c^2(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

$$\text{Minkowski:} ds^2 = -c^2(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

■in Minkowski Spacetime, the line element(spacetime interval) ds are invariant in different inertial reference frame

$$ds^2 = -c^2(dt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

$$= -c^2(dt')^2 + (dx')^2 + (dy')^2 + (dz')^2 = ds'^2$$

Minkowski geometry(2/2)

■the Minkowski Spacetime are flat spacesame as Euclidean spacetime. So Minkowski Spacetime is also called pseudo-Euclidean spacetime

Euclidean Metric: $g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ Minkowski Metric: $\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

■a Lorentz transformation is represented by a matrix $(\Lambda)^{\mu}{}_{\nu}$ that acts on the four-vectorhe in Minkowski Spacetime

$$\chi'^{\mu} = \Lambda^{\mu}{}_{\nu}\chi_{\nu}$$

$$\Rightarrow \text{invariant}: s^2 = \chi^{\mu}\chi_{\mu} = \eta_{\mu\mu}\chi^{\mu}\chi^{\mu} = constant$$

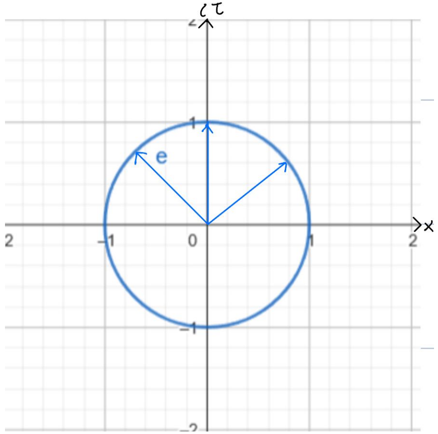
$$\Rightarrow (\Lambda^{\rho}{}_{\nu})^{-1} = \Lambda_{\nu}{}^{\rho} = \eta_{\mu\nu}\Lambda^{\rho\mu} = \eta_{\mu\nu}\eta^{\rho\sigma}\Lambda^{\mu}{}_{\sigma}$$

Minkowski spacetime(1/3)

■the incident with same spacetime interval will drow a hyperbola in Minkowski Spacetime, and the asymptotic line is the worldline of photon

Euclidean

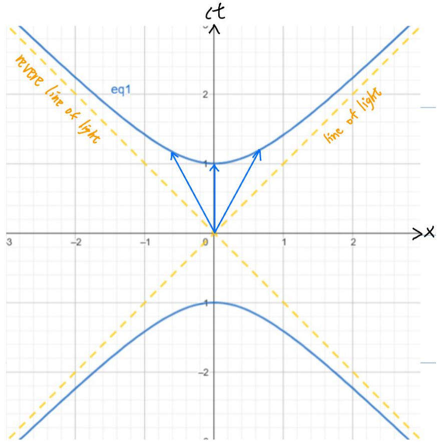
$[ds^2 = c^2(d\tau)^2 + (dx)^2 = a > 0]$



$$t \to i\tau \Leftrightarrow t = e^{i\frac{\pi}{2}}\tau$$

Minkowski

$[ds^2 = -c^2(dt)^2 + (dx)^2 = a < 0]$



■Wick Rotation: if we rotate the timeline t in Minkowski spacetime $\frac{\pi}{2}$ to the imaginary timeline τ in Euclidean spacetime

Minkowski spacetime(2/3)

■**Lorentz transformation is rotation along the hyperbola in Minkowski spacetime called boost.**

$$\text{Lorentz transformation}: \begin{cases} x' = \dfrac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}} \\ ct' = \dfrac{ct - \frac{v}{c}x}{\sqrt{1-\frac{v^2}{c^2}}} \end{cases}$$

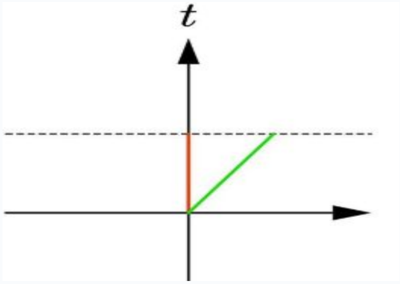
$$\eta = \tanh^{-1}\beta\,,\ \beta = \frac{v}{c}$$

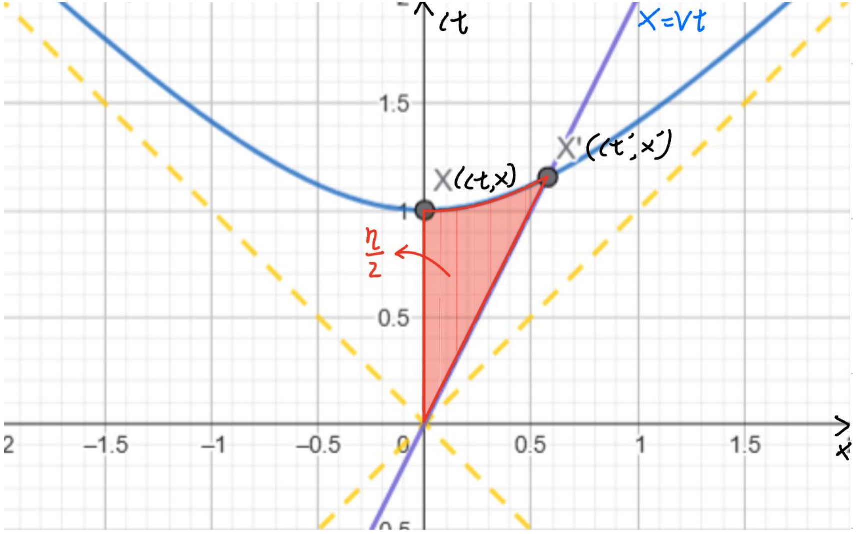
$$\cosh\eta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\ ;\ \sinh\eta = \frac{\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} = \beta\gamma$$

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \begin{bmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix}$$

$$\text{Particle}: \begin{cases} E = \gamma mc^2 = mc^2\cosh\eta \\ |p| = \gamma mv = mc\sinh\eta \end{cases}$$

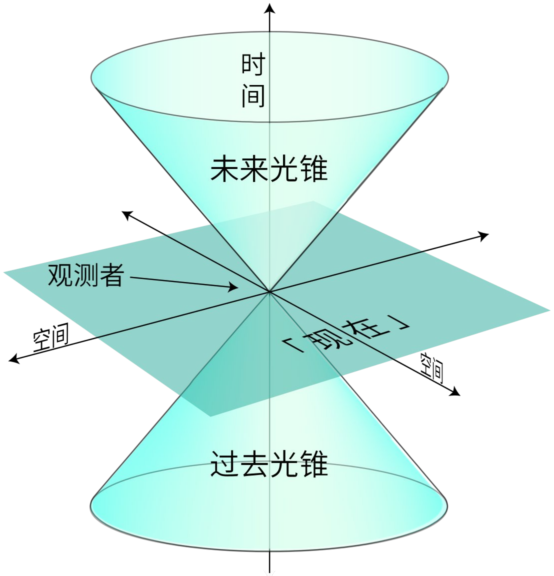
Galilean transformation:





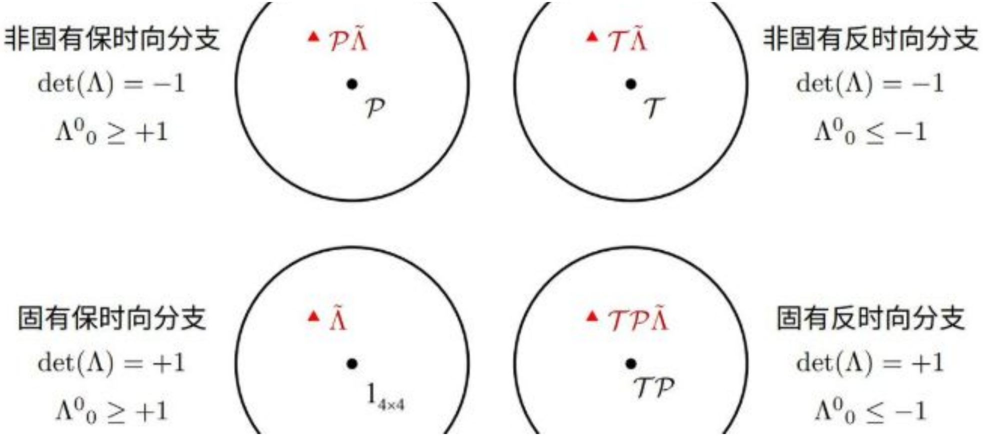
Minkowski spacetime(3/3)

■we call the origin of Minkowski spacetime the present or the observer. In a 2D spacec and 1D time $R^{1,2}$, there will be a curved surface that transmits information at the speed of light called the future light cone in the *+ct* direction; and there will be a curved surface that transmits information at the speed of light called the past light cone in the *-ct*



Lorentz group(1/4)

■**Generalized Lorentz group O(1,3) is a Lie group, which has four subgroup, which represents all Lorentz transformations in Minkowski spacetime**



$$\begin{cases} \Lambda^T \eta \Lambda = \eta \;\Leftrightarrow\; \det(\Lambda) = \pm 1 \\ (\Lambda^0{}_0)^2 = 1 + \sum_{i=1}^{3} \left(\Lambda^i{}_0\right)^2 \geq 1 \end{cases}$$

■**four subgroup can be converted into each other through time reversal *T*=diag(-1,1,1,1) and space reversal *P*=diag(1,-1,-1,-1)**

■**in real world,the Lorentz group is proper orthochronous SO(1,3)**

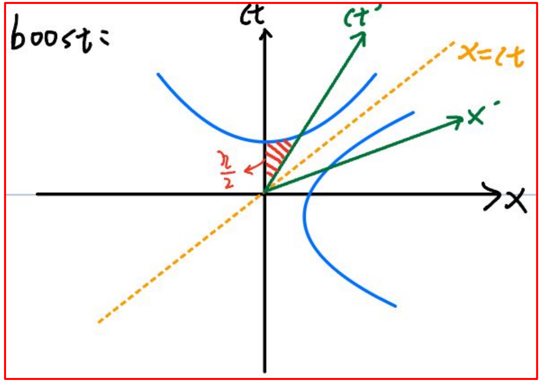
Lorentz group(2/4)

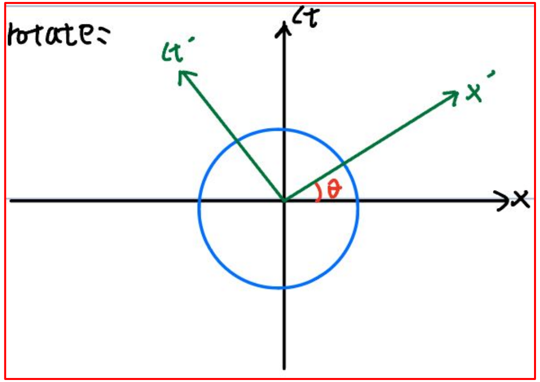
■**Lorentz group SO(1,3) has 3 Lorentz transformation(boost) matrix along xyz-axis and 3 rotation transformation matrix along xyz-axis for a total of 6 degrees of freedom**

$$\Lambda_{boost\,x}=\begin{bmatrix}\cosh\eta & \sinh\eta & 0 & 0\\ \sinh\eta & \cosh\eta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\quad \Lambda_{rotate\,x}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cos\theta & -\sin\theta\\ 0 & 0 & \sin\theta & \cos\theta\end{bmatrix}$$

$$\Lambda_{boost\,y}=\begin{bmatrix}\cosh\eta & 0 & \sinh\eta & 0\\ 0 & 1 & 0 & 0\\ \sinh\eta & 0 & \cosh\eta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\quad \Lambda_{rotate\,y}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & 0 & \sin\theta\\ 0 & 0 & 1 & 0\\ 0 & -\sin\theta & 0 & \cos\theta\end{bmatrix}$$

$$\Lambda_{boost\,z}=\begin{bmatrix}\cosh\eta & 0 & 0 & \sinh\eta\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \sinh\eta & 0 & 0 & \cosh\eta\end{bmatrix}\quad \Lambda_{rotate\,z}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$





Lorentz group(3/4)

■**Lorentz group SO(1,3) then has 3 boost generator *K* and 3 rotation generator *J*. where *K* is symmetry and *J* is antisymmetry**

$$\begin{cases} \Lambda = e^{-i\chi} \\ \chi_i = \theta^i J_i + \eta^i K_i \end{cases}$$

$$(K_x)^\mu{}_\nu = i\frac{d}{d\eta}\Lambda_{boost\ x}|_{\eta=0} = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (J_x)^\mu{}_\nu = i\frac{d}{d\theta}\Lambda_{rotate\ x}|_{\theta=0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$$

$$(K_y)^\mu{}_\nu = i\frac{d}{d\eta}\Lambda_{boost\ y}|_{\eta=0} = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (J_y)^\mu{}_\nu = i\frac{d}{d\theta}\Lambda_{rotate\ y}|_{\theta=0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}$$

$$(K_z)^\mu{}_\nu = i\frac{d}{d\eta}\Lambda_{boost\ z}|_{\eta=0} = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \qquad (J_z)^\mu{}_\nu = i\frac{d}{d\theta}\Lambda_{rotate\ z}|_{\theta=0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow General\ Lorentz\ transformation\ \ \Lambda = \exp\left(-i\vec{J}\cdot\vec{\theta} - i\vec{K}\cdot\vec{\eta}\right)$$

Lorentz group(4/4)

■because *K* is symmetry and *J* is antisymmetry, then we can combine *K*, *J* into generator *M*, and also combine *η*, *θ* into parameters *ω*

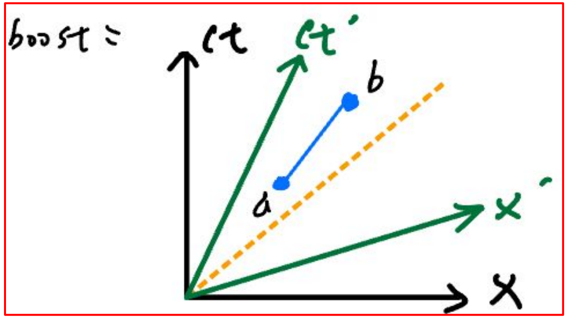
$$\boxed{\chi = \frac{1}{2}\omega_{\mu\nu}M^{\mu\nu}}$$

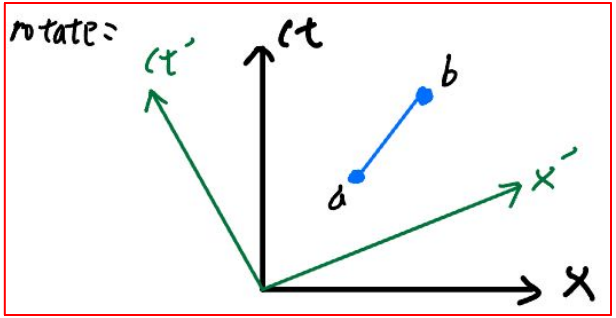
$$\omega = \begin{bmatrix} 0 & \eta^x & \eta^y & \eta^z \\ -\eta^x & 0 & \theta^z & -\theta^y \\ -\eta^y & -\theta^z & 0 & \theta^x \\ -\eta^z & \theta^y & -\theta^x & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & K_x & K_y & K_z \\ -K_x & 0 & J_z & -J_y \\ -K_y & -J_z & 0 & J_x \\ -K_z & J_y & -J_x & 0 \end{bmatrix}$$

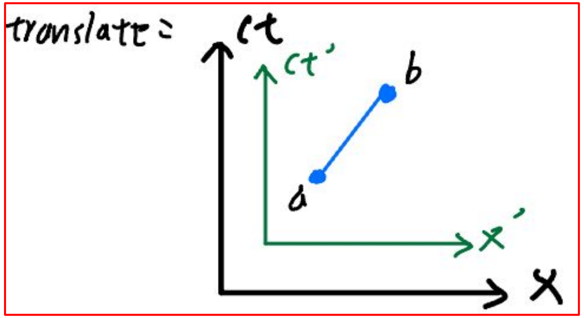
$$\Rightarrow \textit{General Lorentz transformation } \Lambda = e^{-i\chi} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)$$

Poincaré group(1/2)

■now adding Translation in spacetime *a* into general Lorentz transformation *Λ*, then we form a group which ensure that the spacetime interval is is isometry in Minkowski spacetime call Poincaré group







$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu}$$

$$x''^{\mu} = \bar{\Lambda}^{\mu}{}_{\rho}x'^{\rho} + \bar{a}^{\mu}$$

$$= \bar{\Lambda}^{\mu}{}_{\rho}\Lambda^{\mu}{}_{\nu}x^{\nu} + \left(\bar{\Lambda}^{\mu}{}_{\rho}a^{\rho} + \bar{a}^{\mu}\right)$$

$$\Rightarrow T(\bar{\Lambda}, \bar{a})T(\Lambda, a) = T(\bar{\Lambda}\Lambda, \bar{\Lambda}a + \bar{a})$$ [closue]

Poincaré group(2/2)

■**Poincaré group SO(1,3) has has 3 boost generator *K*, 3 rotation generator *J* and 4 spacetime translation generator *P* along txyz-axis for a total of 10 degrees of freedom**

$$\boxed{\begin{cases} \mathrm{R}(\Lambda, \mathrm{a}) = e^{-i\chi} \\ \chi_i = \theta^i J_i + \eta^i K_i + a^i P_i = \frac{1}{2}\omega_{\mu\nu}M^{\mu\nu} + a_\mu P^\mu \end{cases}}$$

$$\Rightarrow \textit{General transformation}\ \ R(\Lambda, \mathrm{a}) = \exp\left(-ia_\mu P^\mu - i\vec{J}\cdot\vec{\theta} - i\vec{K}\cdot\vec{\eta}\right)$$

$$= \exp\left(-ia_\mu P^\mu - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)$$

■**Poincaré group is the full symmetry group of any relativistic field theory. As a result, all elementary particles fall in representations of this group.**

Causal structure(1/2)

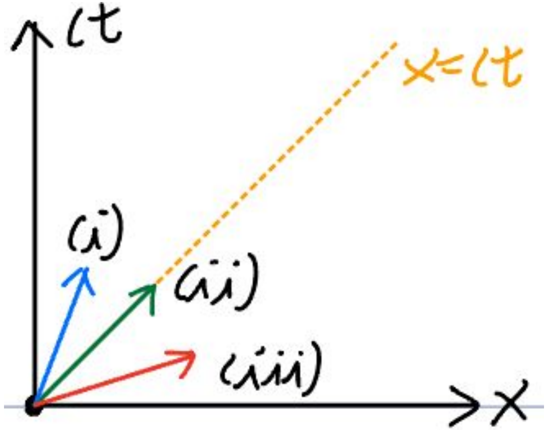
■the nonzero tangent vectors X=(ct,x,y,z) at each point in Minkowski spacetime can be classified into three disjoint types. where timelike and lightlike are called non-spacelike vector.

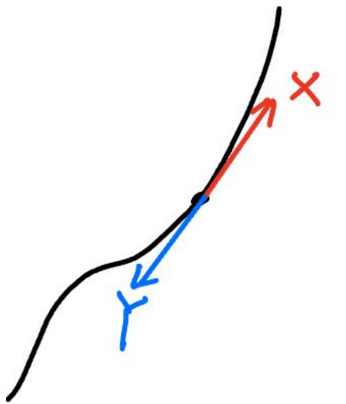
$$\boxed{g(X,X) = g(X,X) = \eta_{\mu\nu}X^\mu X^\nu = X^\mu X_\nu}$$

(i) timelike vector : $g(X,X) < 0$

(ii) lightlike(null) vector : $g(X,X)=0$

(iii) spacelike vector : $g(X,X) > 0$





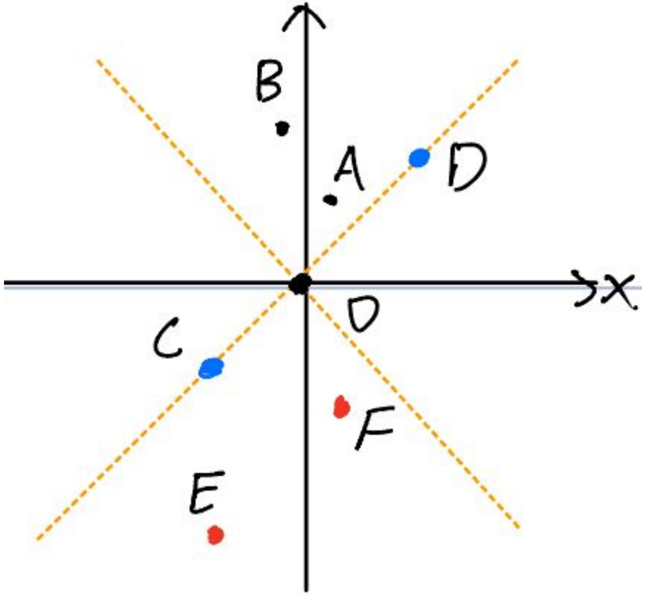
■If X and Y are two timelike tangent vectors at a point we say X and Y are equivalent if $g(X,Y) < 0$. Then we can call one of these future-directed and call the other past-directed, and they has.

Causal structure(2/2)

■**the curve in Minkowski spacetime can be classified into four type. timelike if the tangent vector is timelike at all points in the curve, which also called a world line; lightlike if the tangent vector is null at all points in the curve; spacelike if the tangent vector is spacelike at all points in the curve; causal(non-spacelike) if the tangent vector is timelike or null at all points in the curve.**

$$\begin{cases} \textit{chronological future}: I^{+}(x) = \{y \in R^{4} | x \ll y\} \\ \textit{chronological past}: I^{-}(x) = \{y \in R^{4} | y \ll x\} \end{cases}$$

$$\begin{cases} \textit{causal future}: J^{+}(x) = \{y \in R^{4} | x < y\} \\ \textit{causal past}: J^{-}(x) = \{y \in R^{4} | y < x\} \end{cases}$$



（Ⅱ）Application

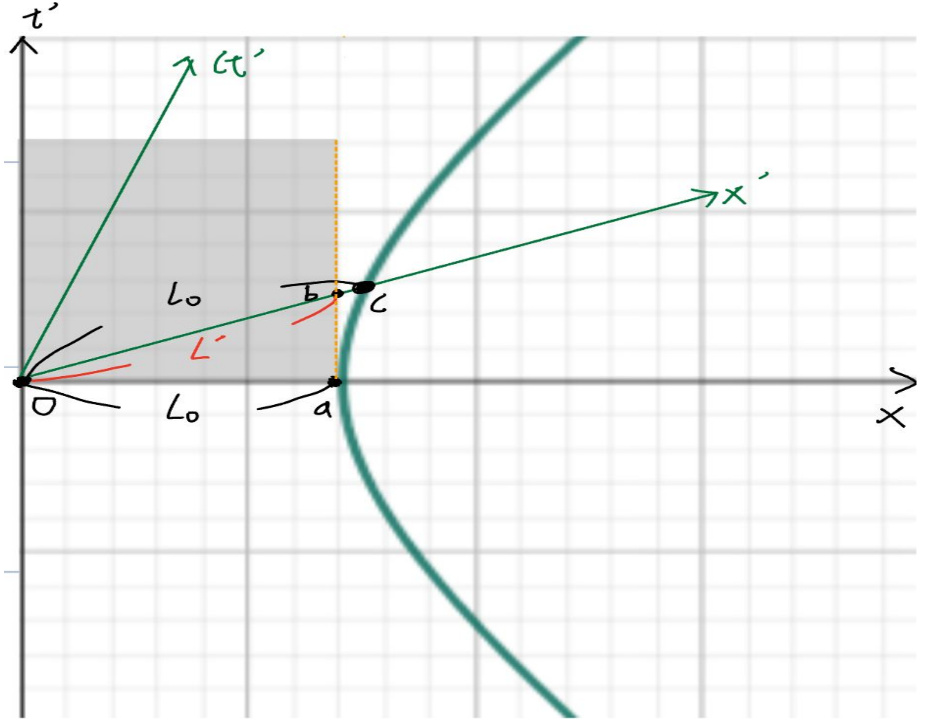
Special relativity(1/2)

■length contraction: a relativistic phenomenon that a moving object's length *L'* is measured to be shorter than its proper length L_0, which is measured in the object's own rest frame.

$$L' = \sqrt{1-\frac{v^2}{c^2}}L_0 = \frac{L_0}{\gamma(v)}$$

■the world line of a particle is a line, and an object is composed of many particles. So the trajectory left by the object should be a surface called the world surface.

$$\overline{ob} < \overline{oc} = \overline{oa}$$



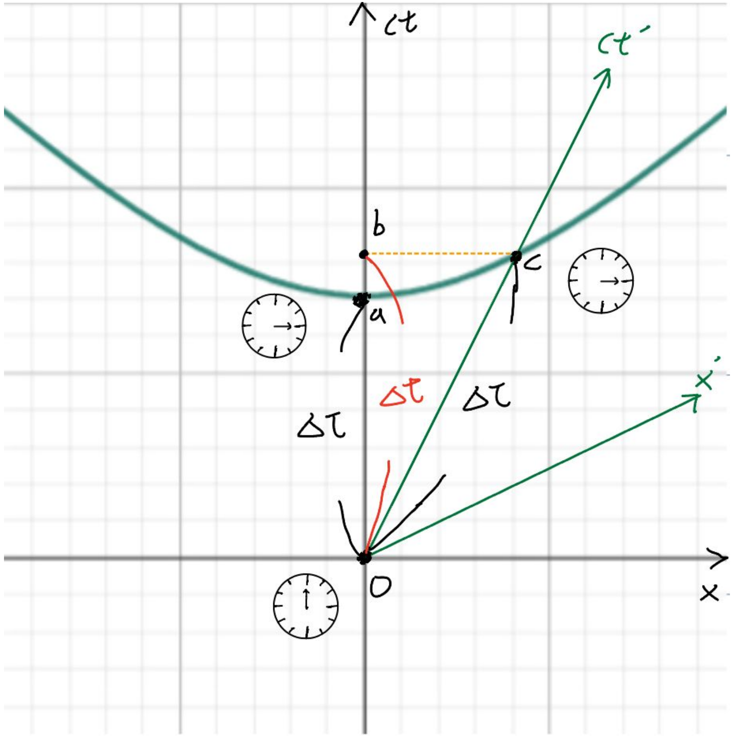
Special relativity(2/2)

■**time dilation：a relativistic phenomenon that the elapsed time *Δt* measured by a moving clock be longer than the elapsed time *Δτ* measured by a rest clock called proper time if the incident is also at rest.**

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma(v)\Delta\tau$$

■**the proper time is equal to the length of the worldline in Minkowski spacetime.That is to say, proper time is the time measured by a clock that remains stationary relative to the particle.**

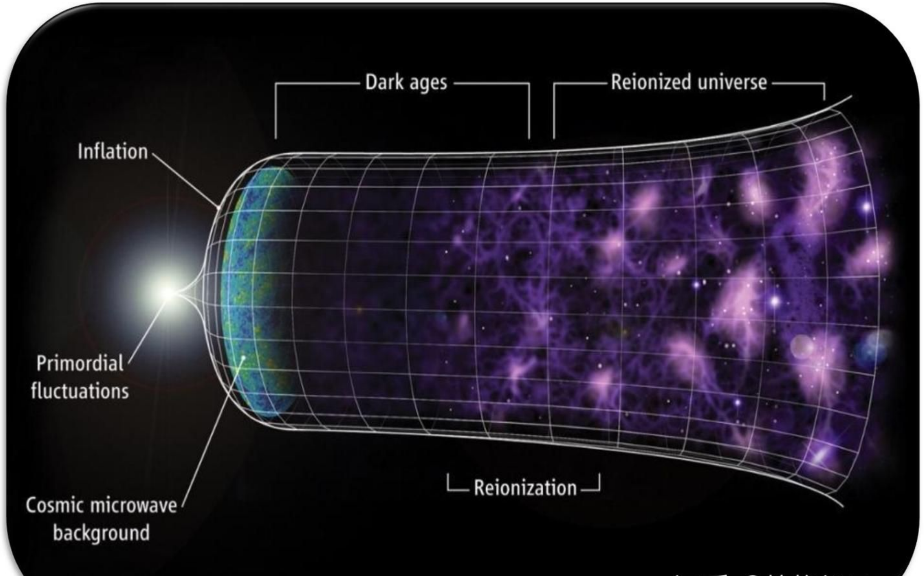
$$\Delta\tau = \frac{1}{c}\int ds = \int\sqrt{1-\frac{v(t)^2}{c^2}}\,dt$$



Big Bang theory

■the universe was in a state of infinite density and temperature before a finite time in the past called a singularity. Therefore, the universe can be represented by a "light cone" (event horizon) on the spacetime diagram, and universe within the event horizon is called the observable universe.

■inflation: Inflation straightens the original light cone of the cosmos, expanding the local areas that were causally connected to the macroscopic scale of the universe. The inflation solve the "horizon" problem in the Big Bang theory.



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