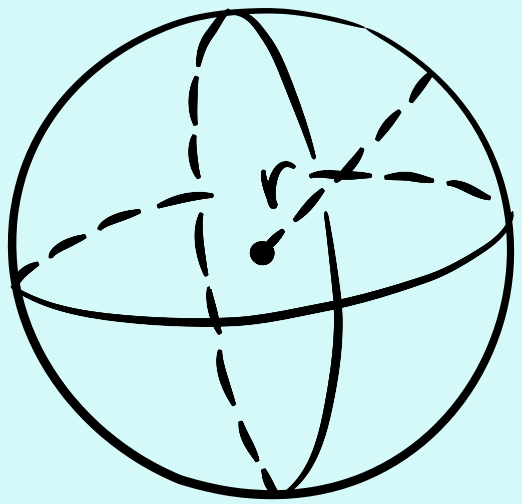
WHAT IS A MANIFOLD

Outline:

1. Topology overview
2. What is manifold
3. Lie group and Lie algebra
4. SO (3) and SU (2)

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

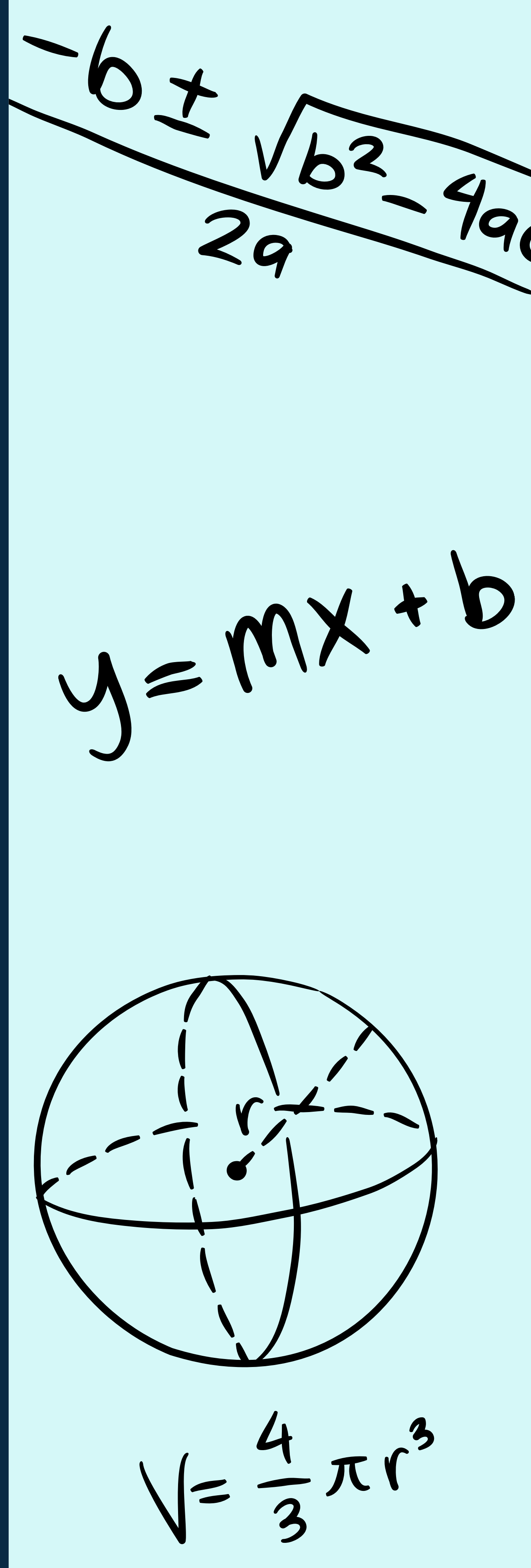
$y = mx + b$



$V = \frac{4}{3}\pi r^3$

Topology:

Before we talk about manifolds, let's talk about some simple topology.

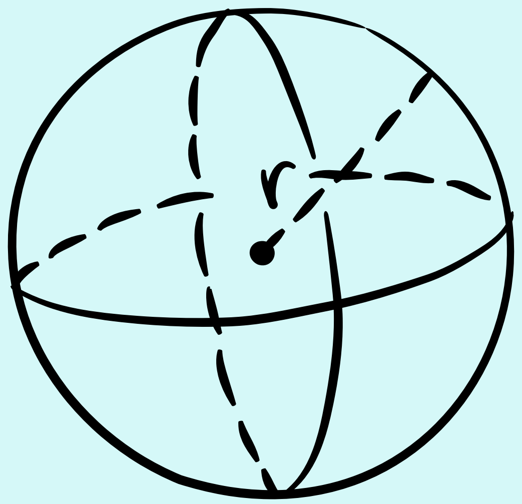


Homeomorphism:

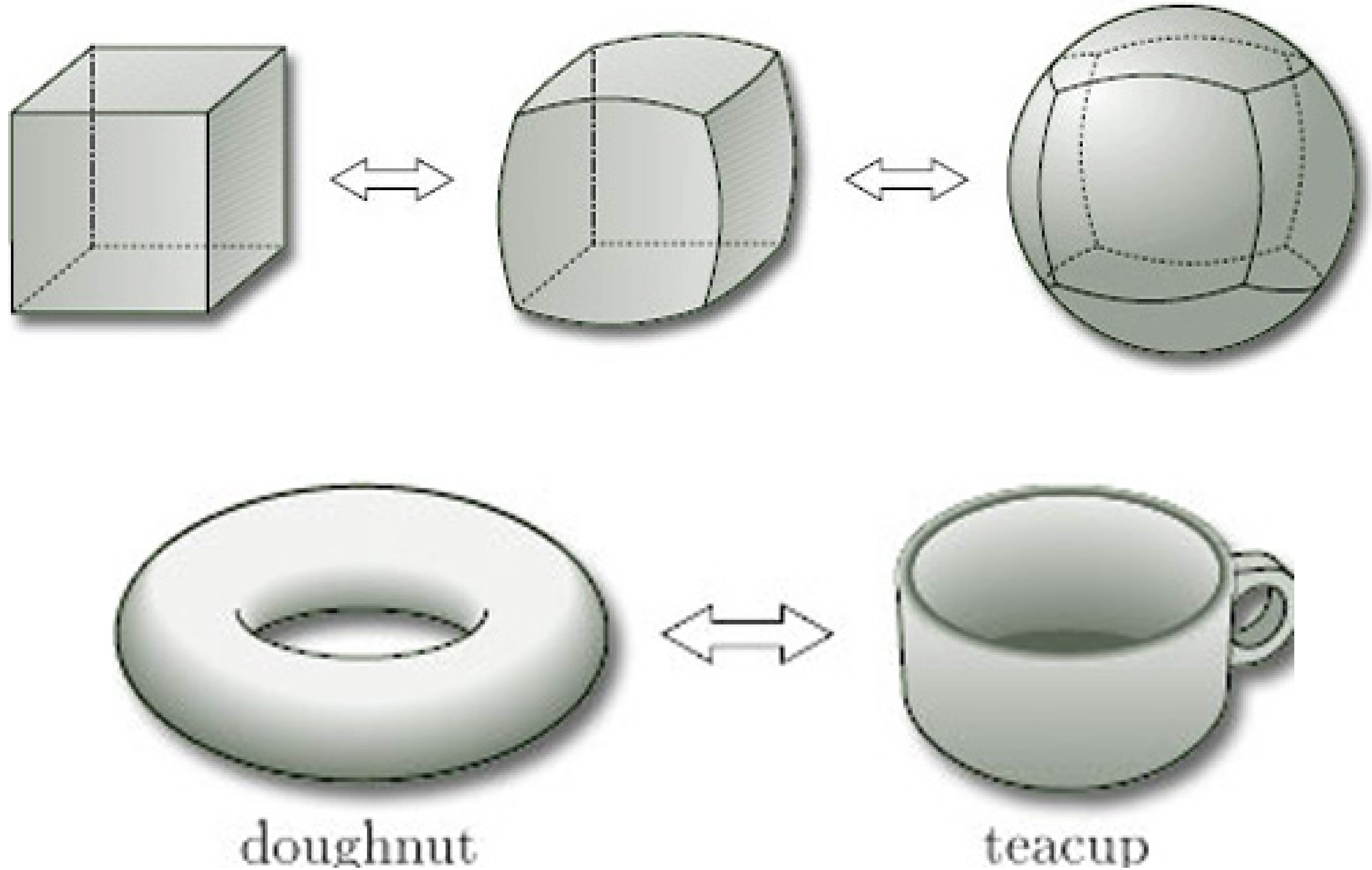
Explain : a homeomorphism means two spaces can be changed into each other by stretching or bending, without cutting or gluing.

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$y = mx + b$$

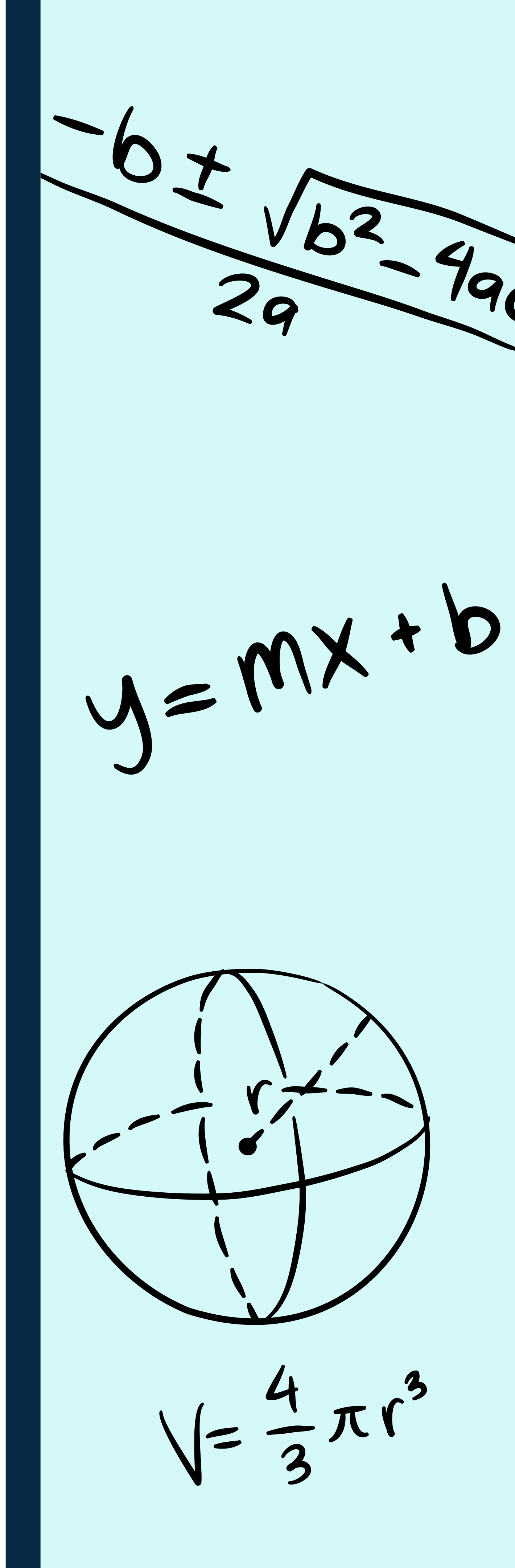


$$V = \frac{4}{3}\pi r^3$$


doughnut
teacup

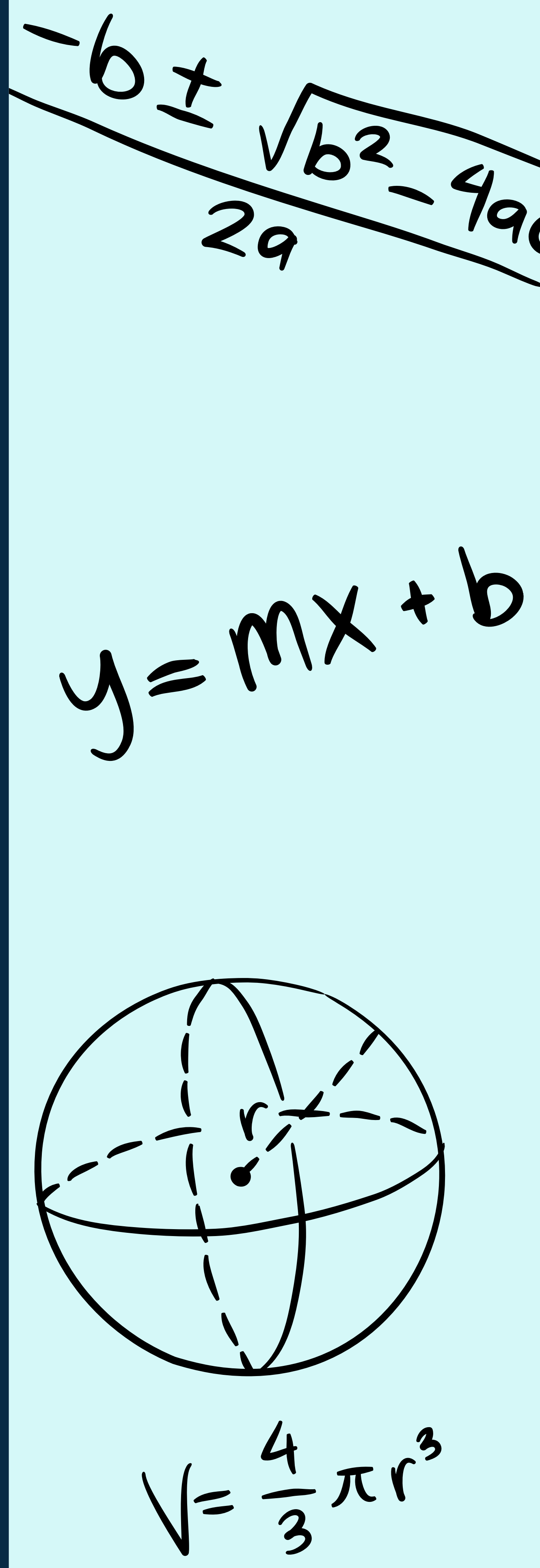
Euclidean space:

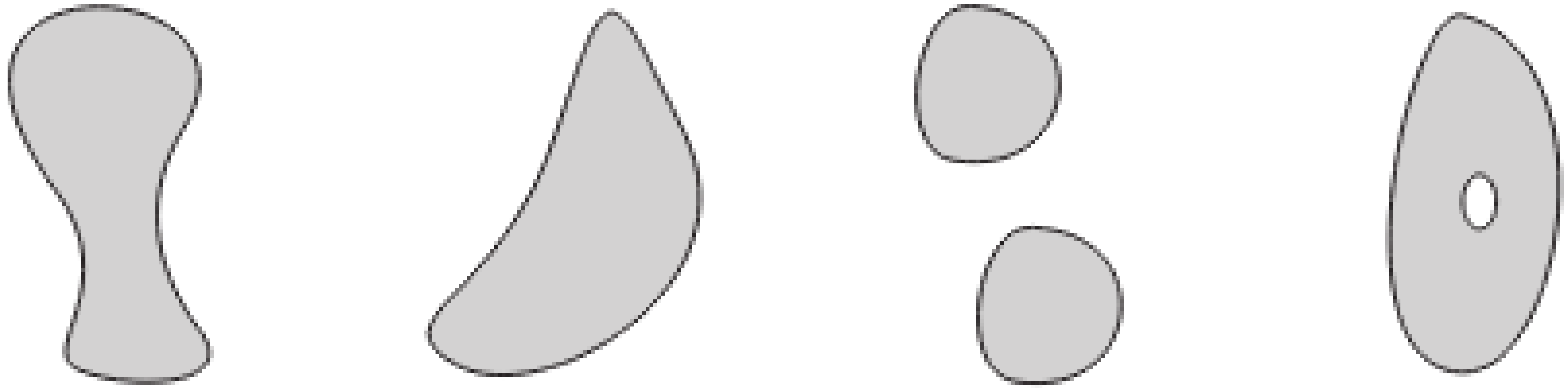
Euclidean space is a space made up of points consisting of n real numbers (**a1,a2,...,an**), denoted by $\mathbf{R}^n$, where the distance between points can be calculated using the familiar square root formula.



Simply connected space:

A space is simply connected if it's all in one piece and has no holes—every loop can shrink to a point.

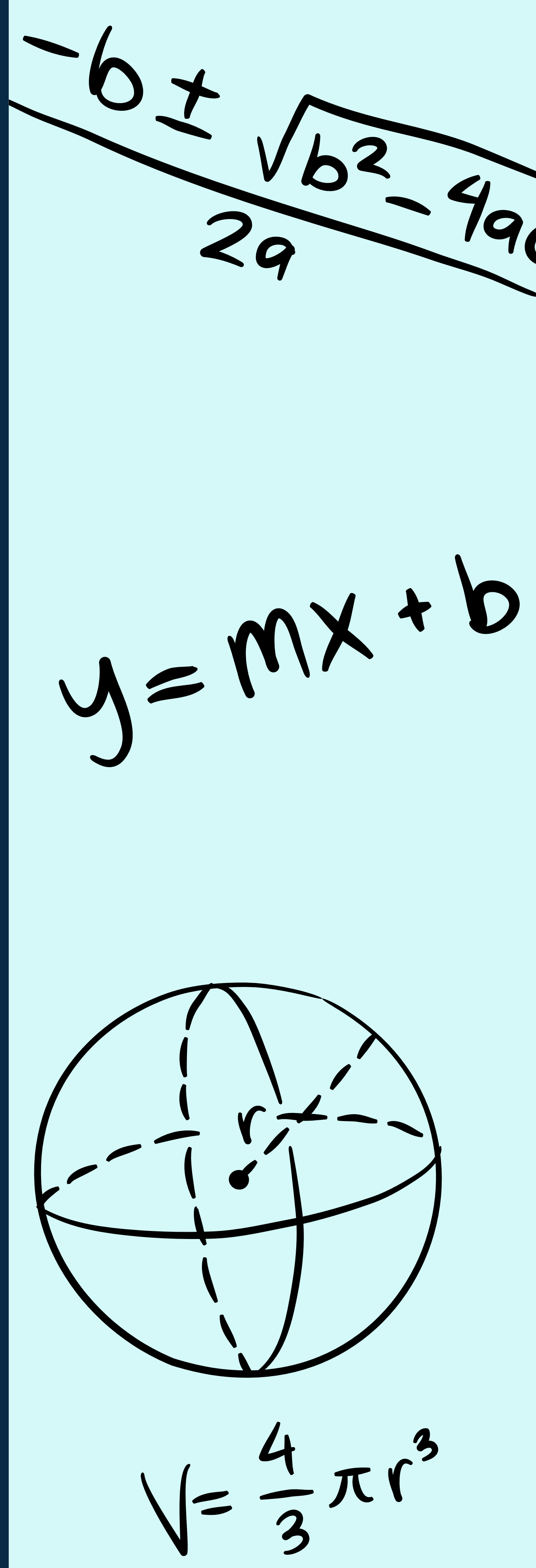


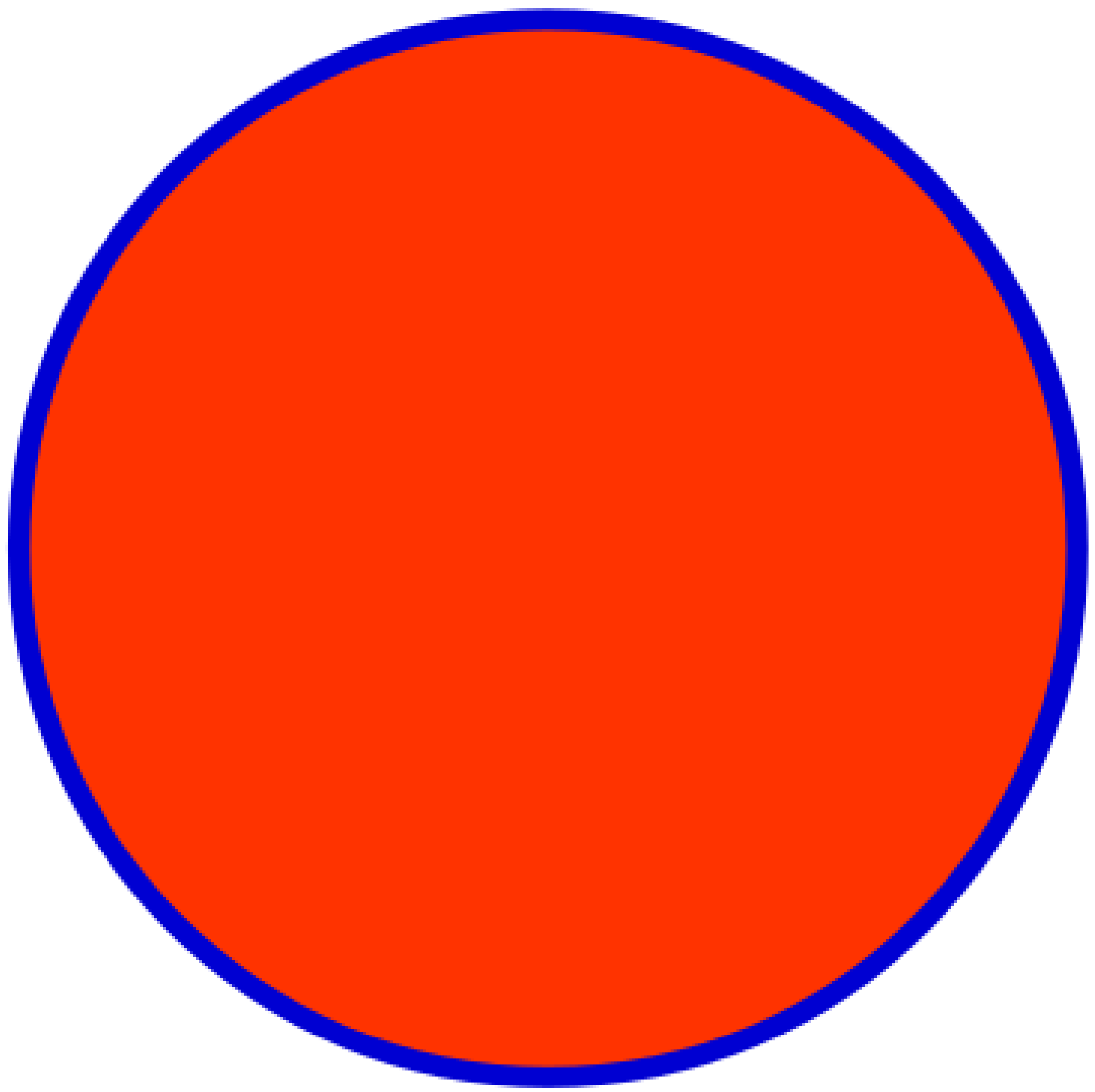


simply connected | *simply connected* | *not simply connected* | *not simply connected*

Open set

An open set is a set where, for every point inside it, you can find a small neighborhood (like a tiny ball) that is completely contained in the set.

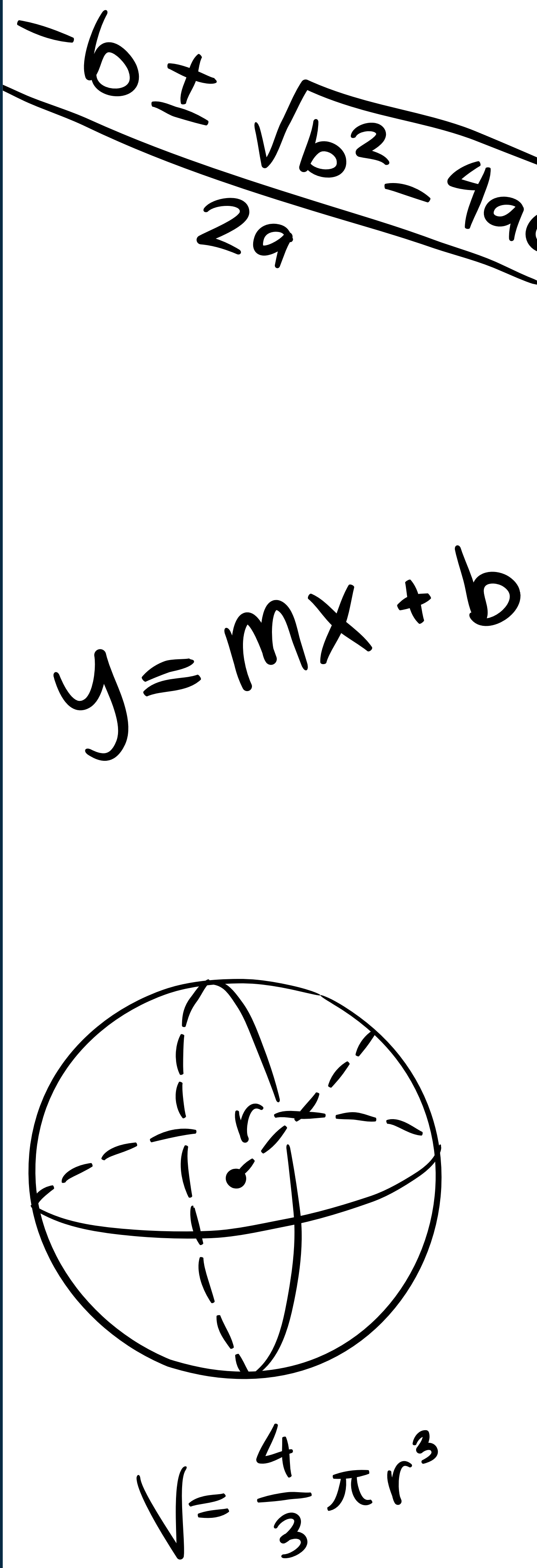




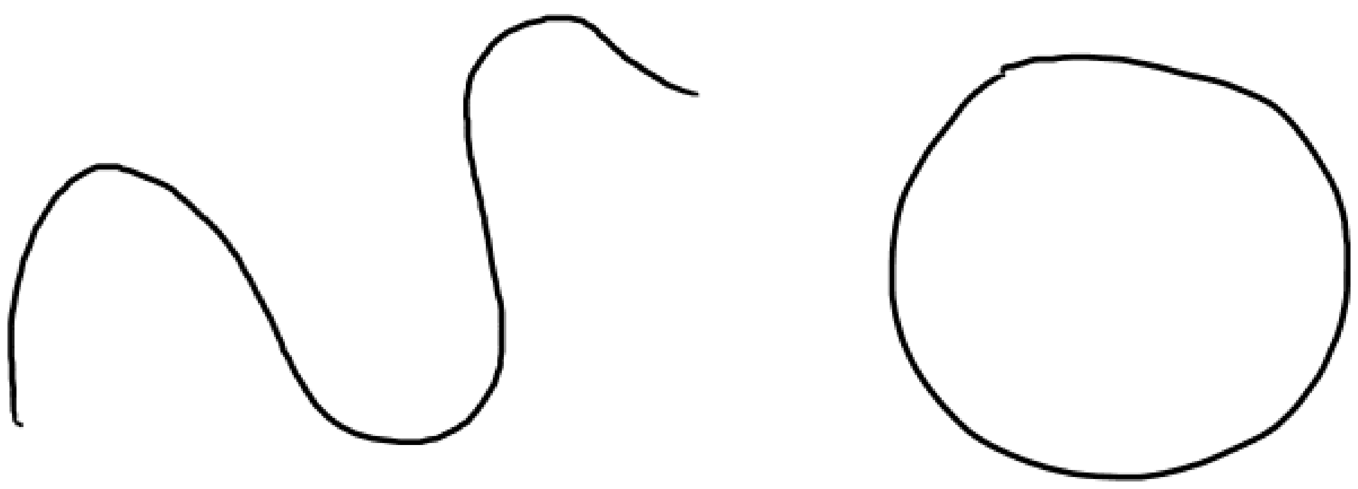
the blue circle represents the set of points (x, y) satisfying $x^2 + y^2 = r^2$. The red disk represents the set of points (x, y) satisfying $x^2 + y^2 < r^2$. The red set is an open set, the blue set is its boundary set, and the union of the red and blue sets is a closed set.

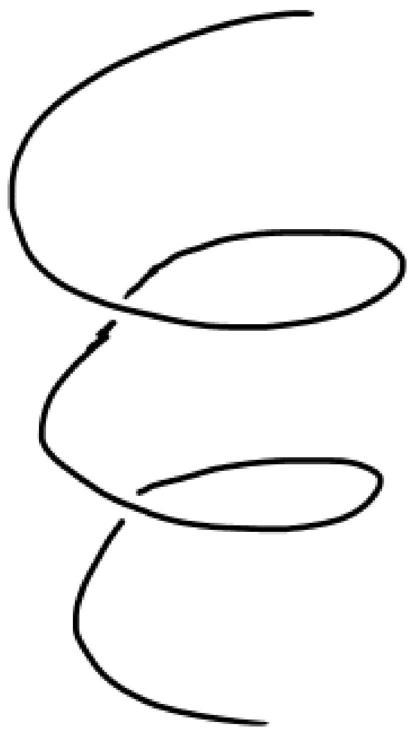
MANIFOLDS

A manifold is a topological space that locally resembles Euclidean space near each point.

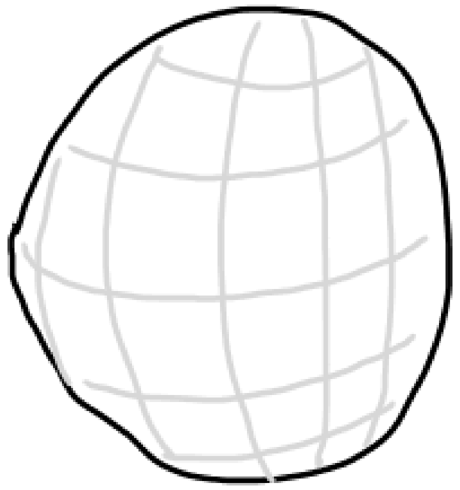


SOME 1D MANIFOLDS:

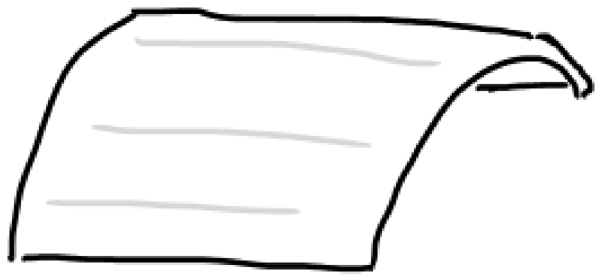


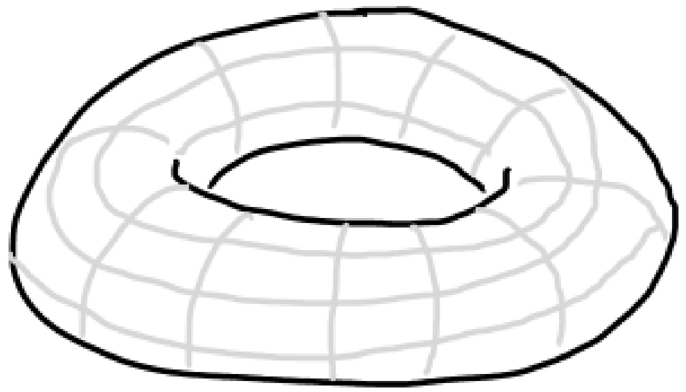


SOME 2D MANIFOLDS:



(SURFACE OF A SPHERE)

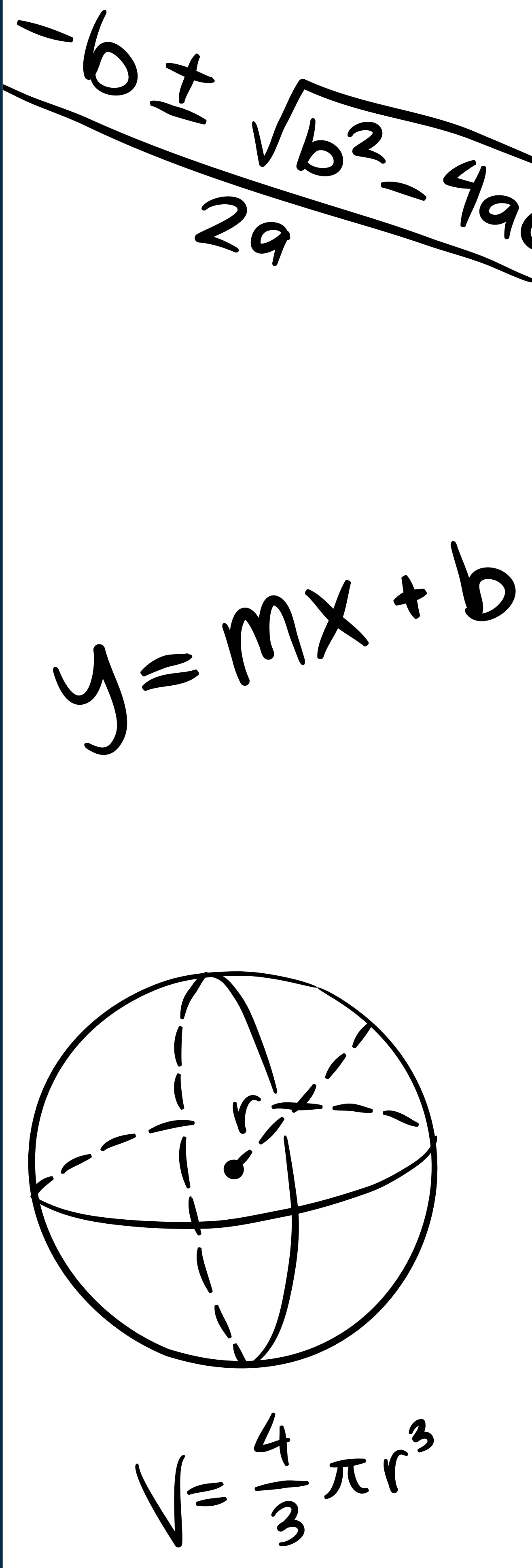




(SURFACE OF A TORUS)

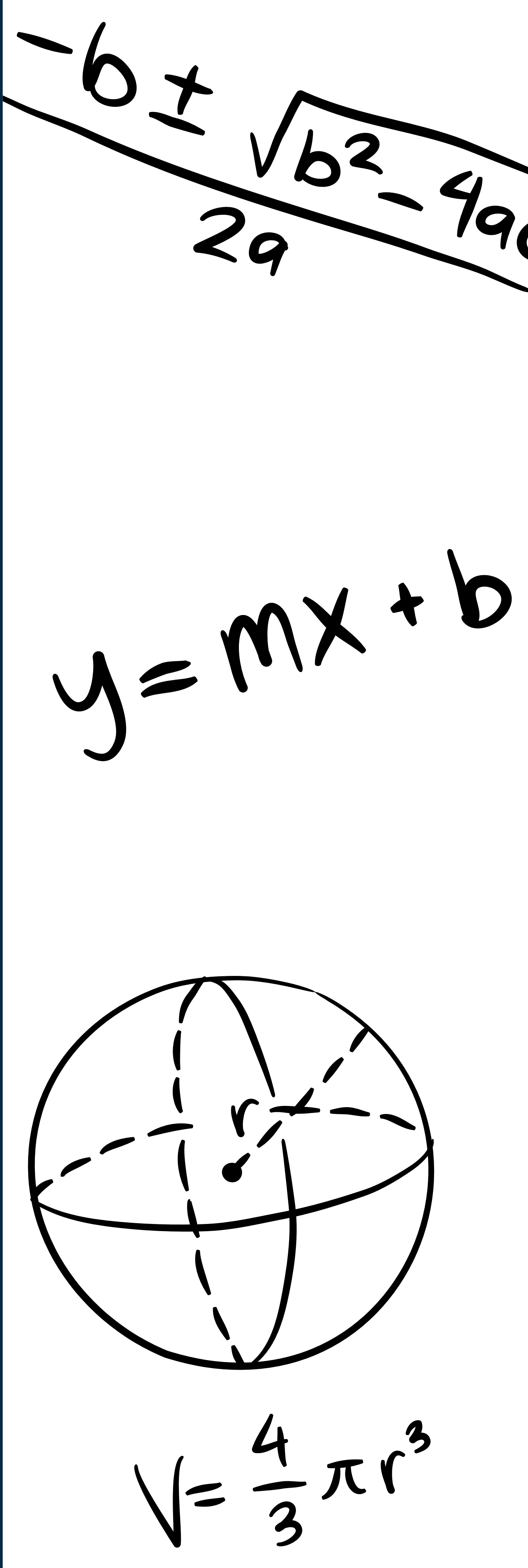
Why study manifolds?

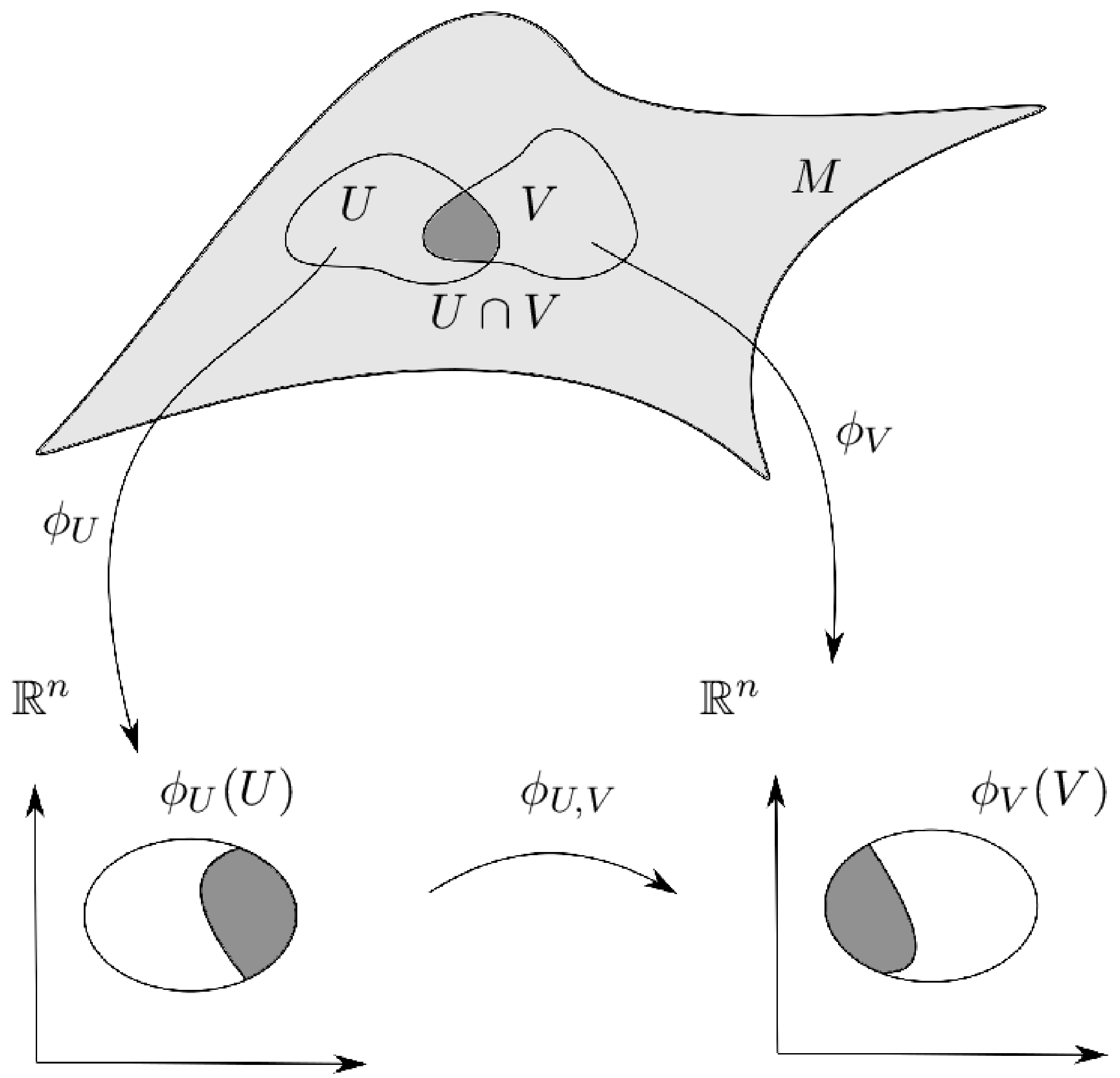
Because in general relativity and field theory, space is not always flat.

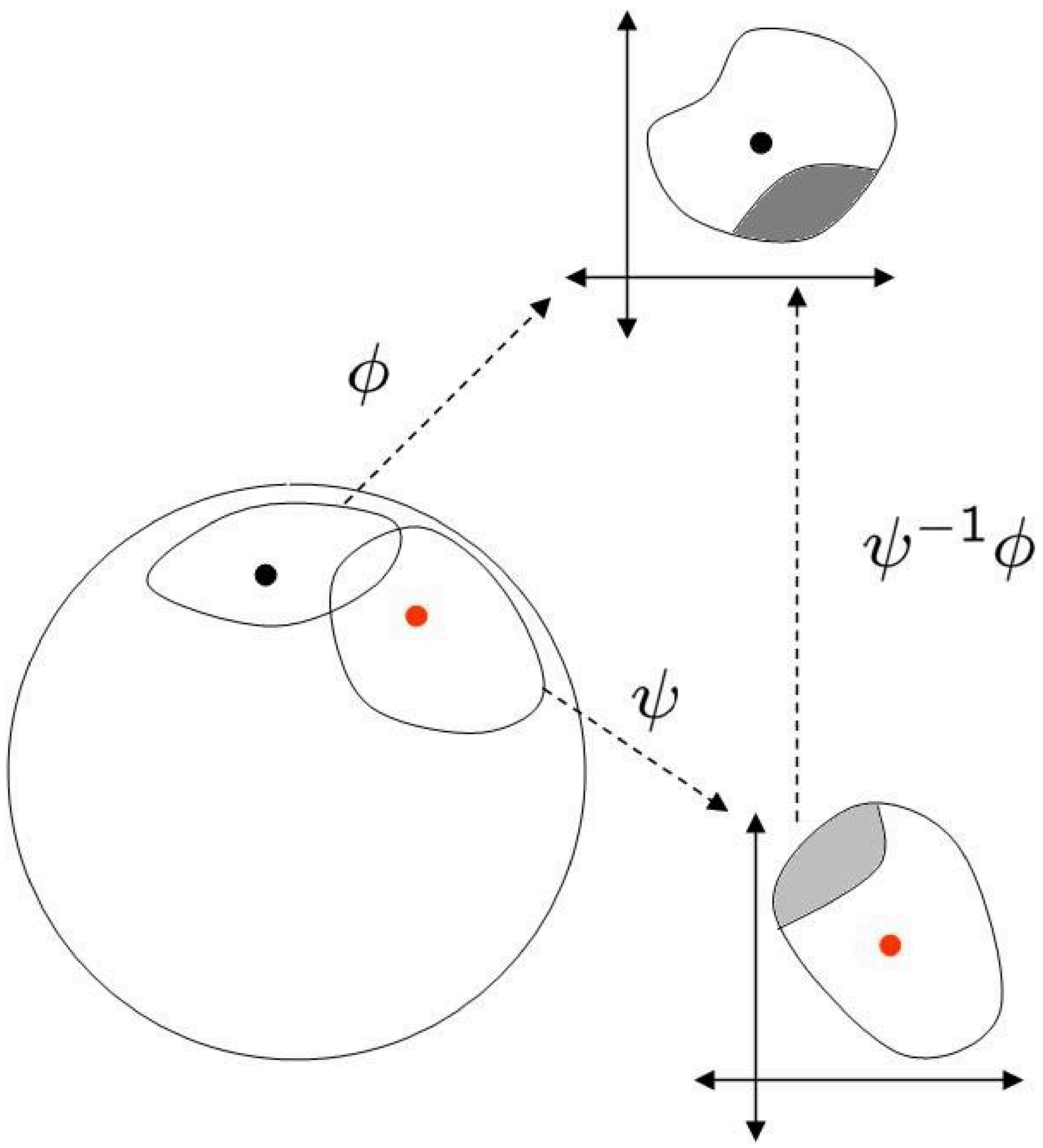


DIFFERENTIABLE MANIFOLD

A smooth manifold is a topological manifold with charts whose transition maps are infinitely differentiable.

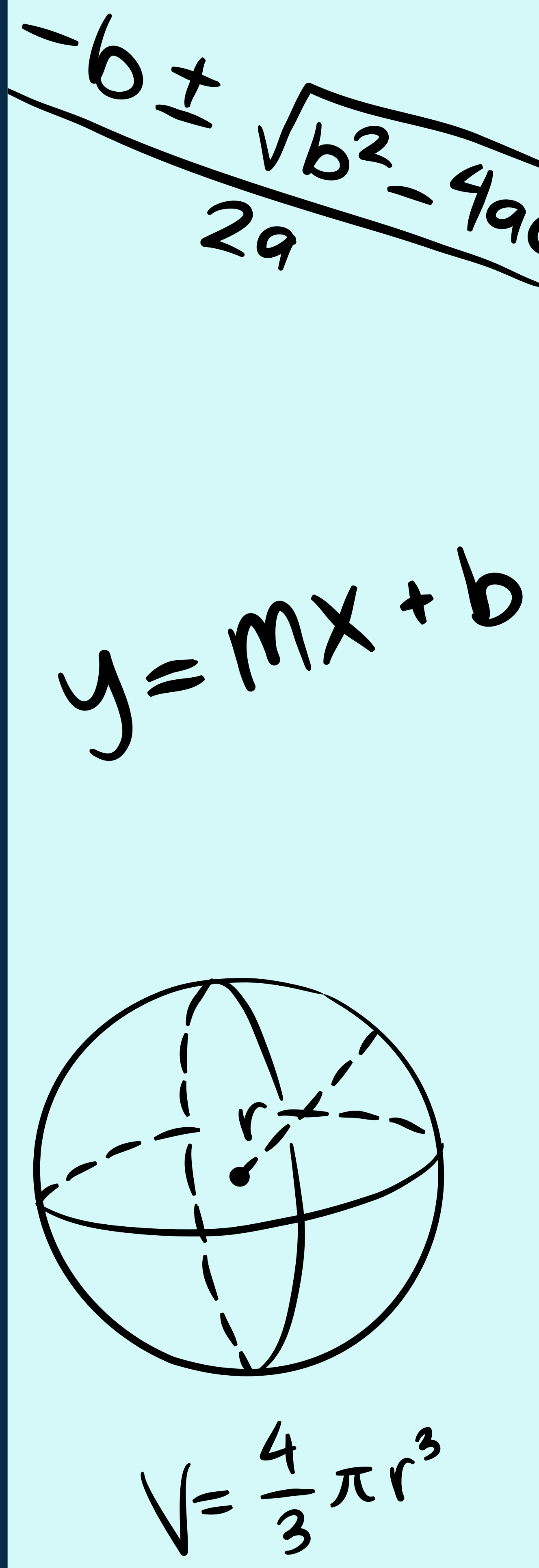



M
U
V
U ∩ V
ϕ_U
ϕ_V
$\mathbb{R}^n$
$\mathbb{R}^n$
$\phi_U(U)$
$\phi_{U,V}$
$\phi_V(V)$


ϕ
$\psi^{-1}\phi$
ψ

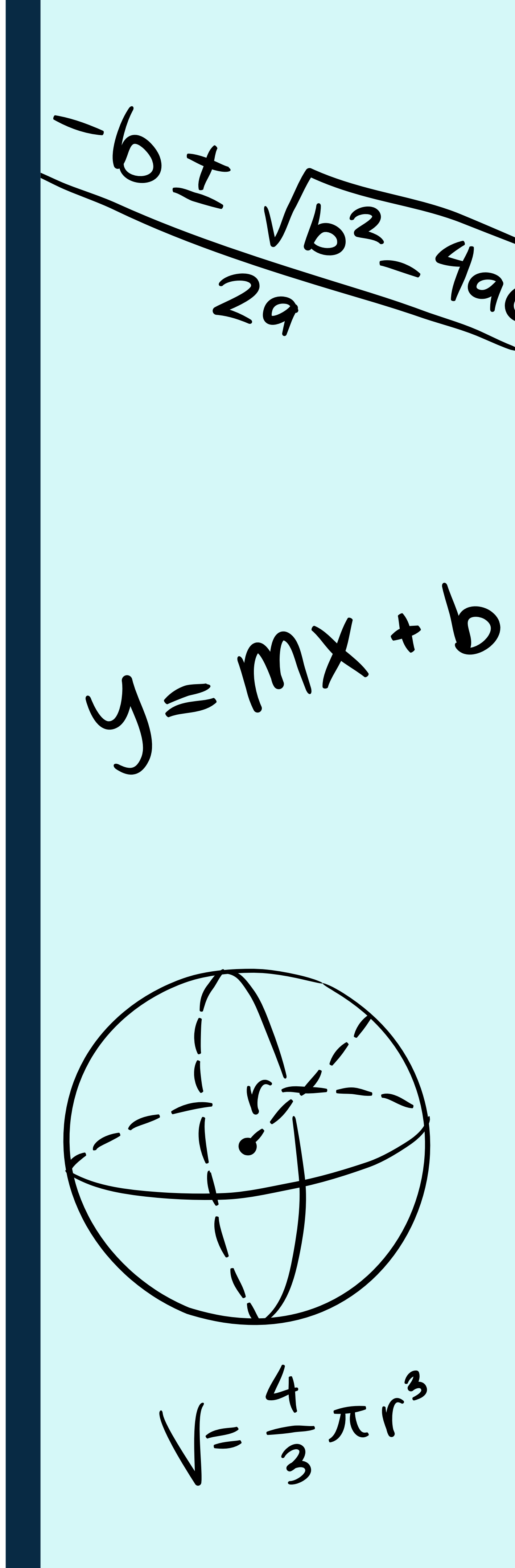
Lie group:

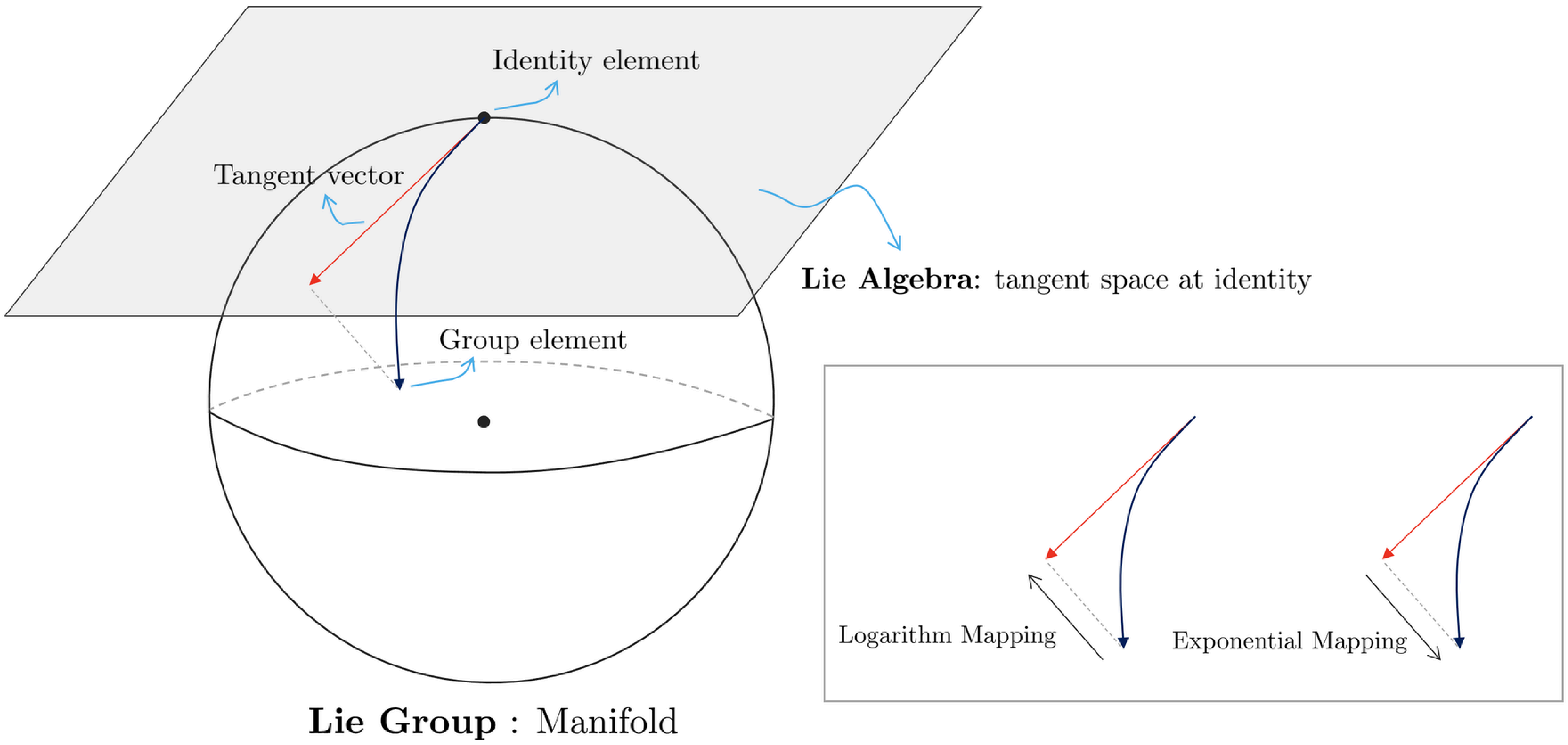
Lie group is a group that is also a differentiable manifold.



Lie algebra:

A Lie algebra describes the local linear structure of a Lie group.
It is a vector space equipped with a binary operation called the Lie bracket.



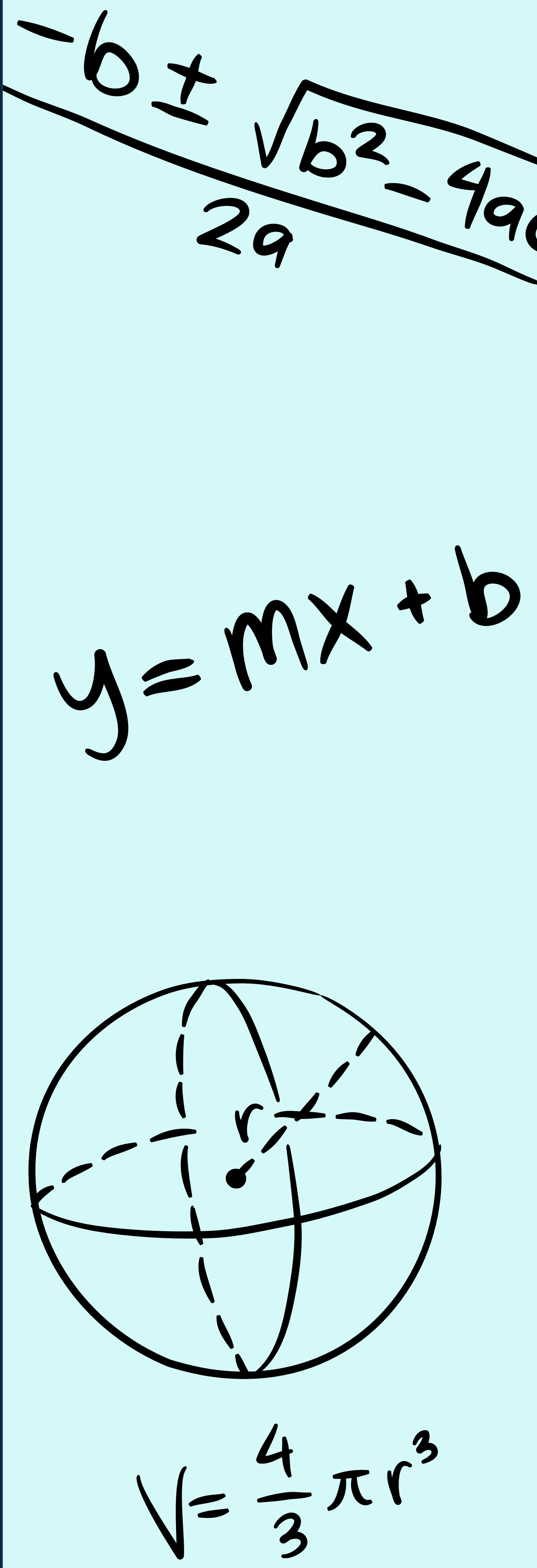

Identity element
Tangent vector
Lie Algebra: tangent space at identity
Group element
Lie Group : Manifold
Logarithm Mapping
Exponential Mapping

3D ROTATION GROUP (SO(3))

SO(3) is the group of all rotations in 3-dimensional space, all operations that rotate an object around the origin without changing its shape.

For example : rotation about the positive z-axis

$$R_z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$



Special unitary group (SU(2))

The special unitary group of degree n, denoted SU(n), is the Lie group of n × n unitary matrices with determinal 1

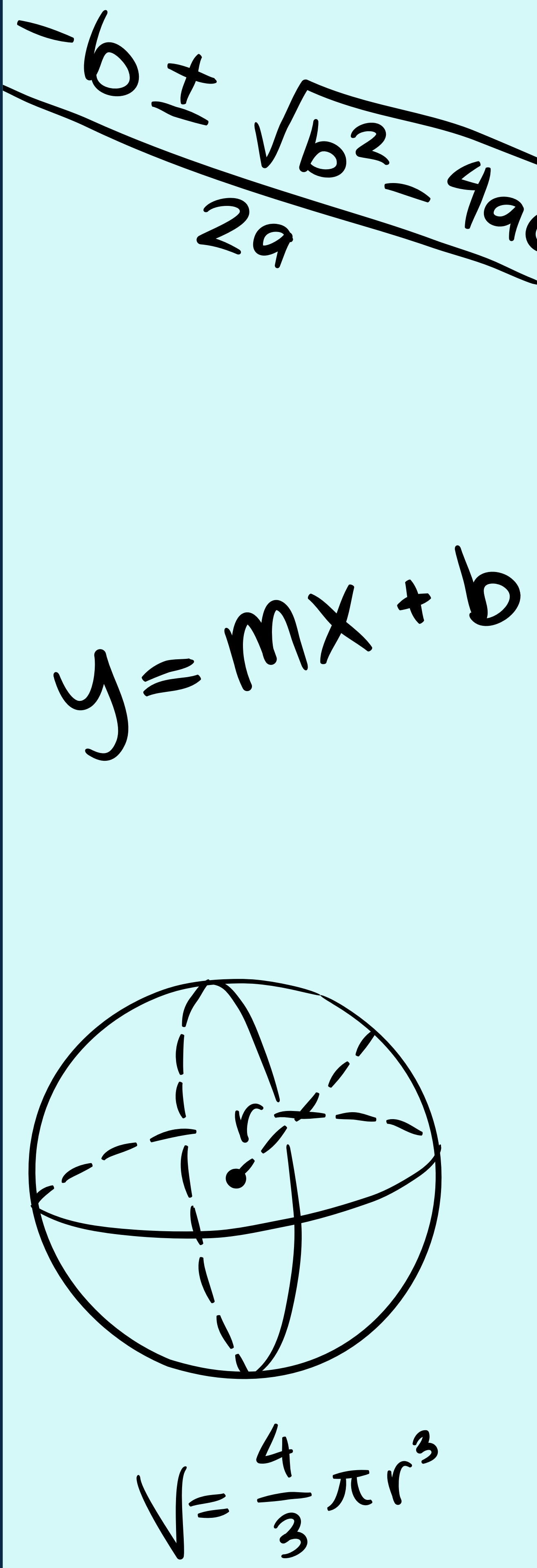
$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \ \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\},$$

where the overline denotes complex conjugation.

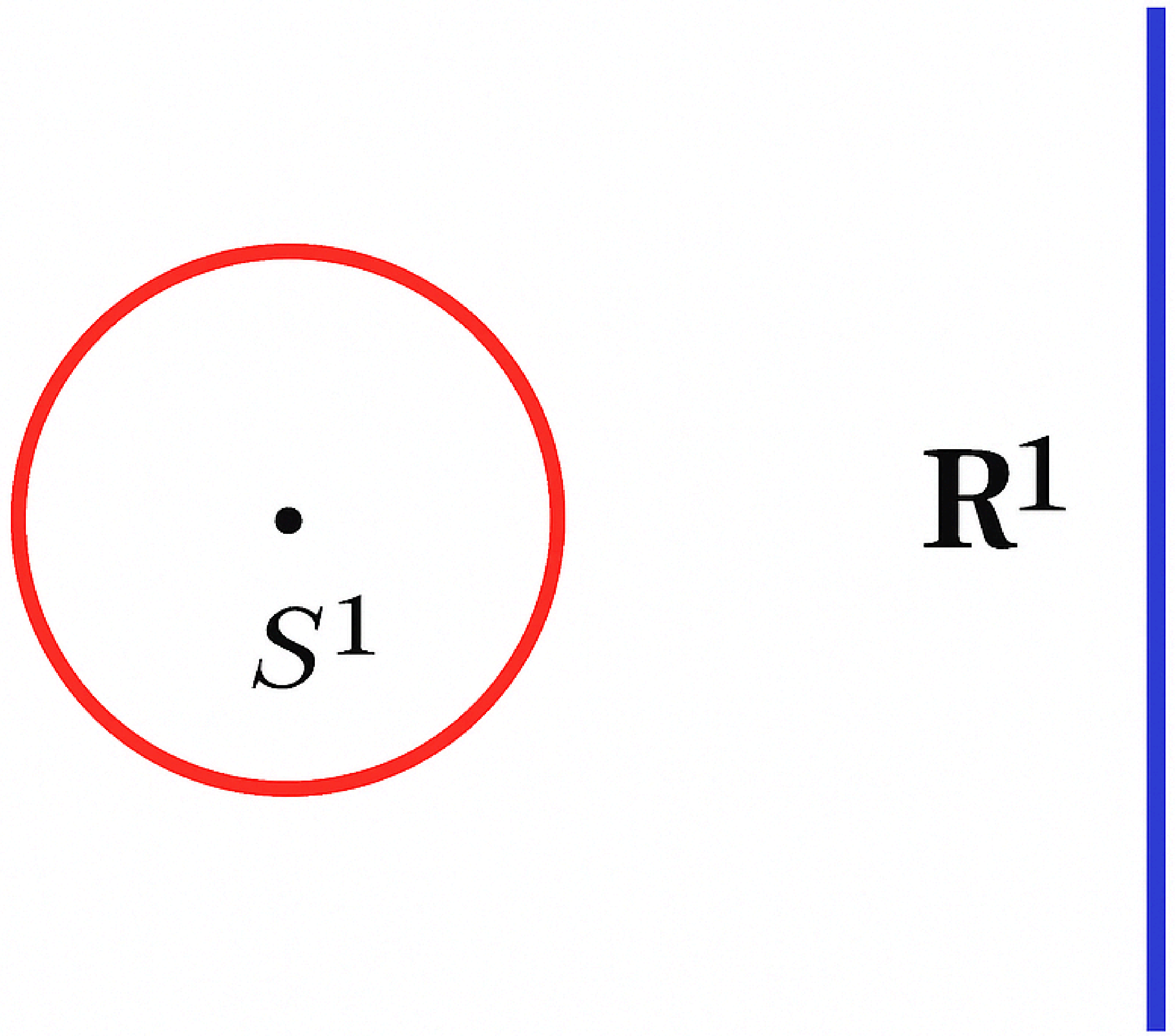
Diffeomorphism with the 3-sphere S^3 [edit]

If we consider α, β as a pair in $\mathbb{C}^2$ where $\alpha = a + bi$ and $\beta = c + di$, then the equation $|\alpha|^2 + |\beta|^2 = 1$

$$a^2 + b^2 + c^2 + d^2 = 1$$



Property	SO(3)	SU(2)
Group Type	Rotation group in 3D space	Special unitary group of 2×2 matrices
Matrix Representation	3×3 real orthogonal matrices	2×2 complex unitary matrices with det = 1
Determinant	det = +1	det = 1
Lie Algebra	so(3)	su(2) ≅ so(3)
Dimension	3	3
Topology	$S^3 / \mathbb{Z}_2$ (3D real projective space)	S^3 (3D surface of 4D ball)
Simply Connected	No	Yes
Physical Application	Classical rotation (e.g., rigid body)	Quantum spin-½ particles
Relationship	Covered by SU(2)	Double cover of SO(3)



S^1
$\mathbf{R}^1$

References:

1. Topology picture 、simple connect 、openset
2. manifold 1 、manifold2
3. Lie group and Lie algebra
4. SO（3）and SU（2）
5. Image generated by Chatgpt

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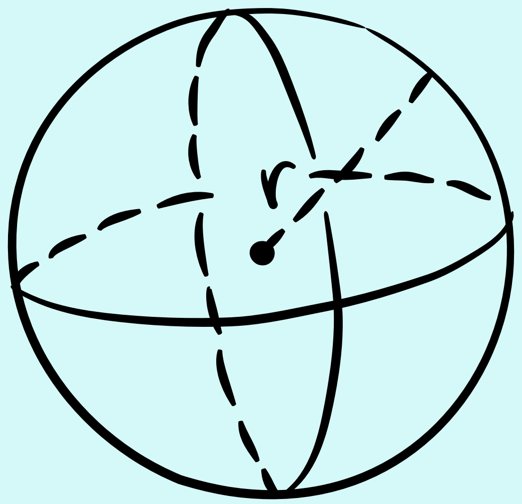
$$y = mx + b$$



$$V = \frac{4}{3}\pi r^3$$