Covariant Derivative & Christoffel Symbol

物理系 大二 陳柏霖 112022112

Simply Derivative of Tensors

- Let T be a contravariant tensor of rank 1, $T = T^i$
- Let the x^icoordinate system $[x^1$,$x^2, \dots, x^n]$ be a coordinate system in $\mathbb{R}^n$,where 1$\leq i \leq n$
- Let $\tilde{x}^i$ be a new coordinate system

$$\tilde{T}^i = T^r \frac{\partial \tilde{x}^i}{\partial x^r}$$

If we differentiate the whole sentence w.r.t $\tilde{x}^k$

$$\frac{\partial \tilde{T}^i}{\partial \tilde{x}^k} = \frac{\partial}{\partial \tilde{x}^k}\left(T^r \frac{\partial \tilde{x}^i}{\partial x^r}\right) = \frac{\partial T^r}{\partial \tilde{x}^k}\frac{\partial \tilde{x}^i}{\partial x^r} + T^r \frac{\partial}{\partial \tilde{x}^k}\left(\frac{\partial \tilde{x}^i}{\partial x^r}\right)$$

Using chain rule and Einstein notation

$$\frac{\partial \tilde{T}^i}{\partial \tilde{x}^k} = \frac{\partial x^s}{\partial \tilde{x}^k}\frac{\partial T^r}{\partial x^s}\frac{\partial \tilde{x}^i}{\partial x^r} + T^r \frac{\partial x^s}{\partial \tilde{x}^k}\frac{\partial}{\partial x^s}\left(\frac{\partial \tilde{x}^i}{\partial x^r}\right)$$

$$= \frac{\partial T^r}{\partial x^s}\frac{\partial x^s}{\partial \tilde{x}^k}\frac{\partial \tilde{x}^i}{\partial x^r} + T^r \frac{\partial x^s}{\partial \tilde{x}^k}\frac{\partial^2 \tilde{x}^i}{\partial x^s \partial x^r}$$

$$\frac{\partial \tilde{T}^i}{\partial \tilde{x}^k} = \frac{\partial T^r}{\partial x^s}\frac{\partial x^s}{\partial \tilde{x}^k}\frac{\partial \tilde{x}^i}{\partial x^r} + T^r \frac{\partial x^s}{\partial \tilde{x}^k}\frac{\partial^2 \tilde{x}^i}{\partial x^s \partial x^r}$$

Isn't it familiar to the definition of a (1,1) type tensor?

$$\tilde{A}_q{}^p = A_n{}^m \frac{\partial \tilde{x}^p}{\partial x^m}\frac{\partial x^n}{\partial \tilde{x}^q}$$

That if $\frac{\partial^2 \tilde{\mathrm{x}}^{\mathrm{i}}}{\partial \mathrm{x}^{\mathrm{s}} \partial \mathrm{x}^{\mathrm{r}}} = 0$, then $\frac{\partial \mathrm{T}^{\mathrm{i}}}{\partial \mathrm{x}^{\mathrm{k}}}$ is indeed a tensor

$$\frac{\partial \tilde{T}^i}{\partial \tilde{x}^k} = \frac{\partial T^r}{\partial x^s}\frac{\partial x^s}{\partial \tilde{x}^k}\frac{\partial \tilde{x}^i}{\partial x^r}$$

Special Case ($\frac{\partial^2 \tilde{x}^i}{\partial x^s \partial x^r} = 0$)

This will happen in an affine coordinate transform:

- When $\tilde{x}^i = kx^i$, k is a constant
- When $\tilde{x}^i = a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$

But what about curvilinear coordinate transformations?

In curved space or non-Cartesian coordinates, $\frac{\partial \mathrm{T}^i}{\partial \mathrm{x}^\mathrm{k}}$ is no longer a tensor!

Covariant Derivative

To make the derivative of T^i being a tensor in curved space, we need to add a new term

$$\nabla_k T^i \equiv \frac{\partial T^i}{\partial x^k} + T^s \Gamma_{sk}{}^i$$

∇_k is the covariant derivative, which satisfy $\tilde{\nabla}_k \tilde{T}^i = \nabla_s T^r \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \tilde{x}^k}$

Christoffel Symbol

- The first kind:

$$\Gamma_{\ ijk} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ki} - \frac{\partial}{\partial x^k} g_{ij}\right)$$

- The second kind:

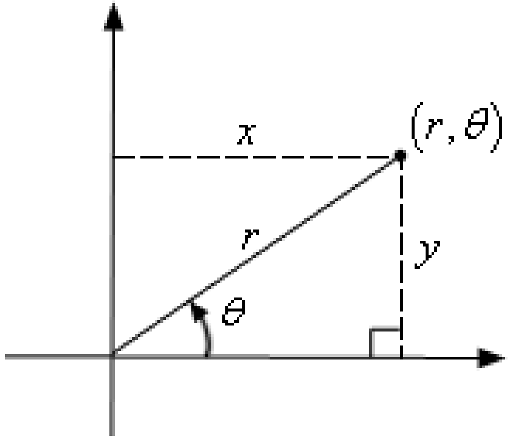
$$\Gamma_{jk}{}^{i} = g^{ir}\Gamma_{\ jkr}$$

Noting that $g_{ij} = \vec{e}_i \cdot \vec{e}_j$ is the metric tensor

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ki} - \frac{\partial}{\partial x^k} g_{ij}\right) \;\; ; \;\; {\Gamma_{jk}}^{i} = g^{ir}\Gamma_{jkr}$$

- $\Gamma_{ijk} = \Gamma_{jik}$; ${\Gamma_{jk}}^{i} = {\Gamma_{kj}}^{i}$
- The Christoffel Symbols all become 0 if and only if g_{ij} are constant

Eg, If we're in 2D polar coordinate, g $= \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$; $\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} r \\ \theta \end{pmatrix}$



Since i,j,k =1 or 2, there will be 8 possible Γ_{ijk}
But by symmetry, we only need to consider 6 possibilities

1D-1; 2D-6; 3D-18; 4D-40; ND-N^2(N+1)/2

Derive Covariant Derivative

$$\tilde{g}_{ij} = g_{rs} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j}$$

$$\frac{\partial \tilde{g}_{ij}}{\partial \tilde{x}^k} = \frac{\partial x^t}{\partial \tilde{x}^k} \frac{g_{rs}}{\partial x^t} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} + g_{rs} \left(\frac{\partial^2 x^r}{\partial \tilde{x}^k \partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} + \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial^2 x^s}{\partial \tilde{x}^k \partial \tilde{x}^j} \right)$$

$\frac{g_{ij}}{\partial x^k}$ is not a tensor either, unless we’re in affine coordinate transformation.

Note, $\Gamma_{\ ijk} = \frac{1}{2} (\frac{\partial}{\partial x^i} g_{jk} + \frac{\partial}{\partial x^j} g_{ki} - \frac{\partial}{\partial x^k} g_{ij})$

$$\tilde{\Gamma}_{ijk} = \Gamma_{rst} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial x^t}{\partial \tilde{x}^k} + g_{rs} \frac{\partial^2 x^r}{\partial \tilde{x}^j \partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^k}$$

$$\tilde{\Gamma}_{jk}{}^{i} = \Gamma_{rs}{}^{u} \frac{\partial \tilde{\mathrm{x}}^{\mathrm{i}}}{\partial \mathrm{x}^{\mathrm{u}}} \frac{\partial x^r}{\partial \tilde{x}^j} \frac{\partial x^s}{\partial \tilde{x}^k} + \frac{\partial^2 x^r}{\partial \tilde{x}^j \partial \tilde{x}^k} \frac{\partial \tilde{\mathrm{x}}^{\mathrm{i}}}{\partial \mathrm{x}^{\mathrm{r}}}$$

$$\vdots$$

$$\frac{\partial \tilde{T}^i}{\partial \tilde{x}^k} + \tilde{T}^s \Gamma_{sk}{}^{i} = \left(\frac{\partial T^r}{\partial x^s} + T^t \Gamma_{ts}{}^{r} \right) \frac{\partial \tilde{\mathrm{x}}^{\mathrm{i}}}{\partial \mathrm{x}^{\mathrm{r}}} \frac{\partial x^s}{\partial \tilde{x}^k}$$

In general:

- The covariant derivative of a (1,0) tensor

$$\nabla_k T^i \equiv \frac{\partial T^i}{\partial x^k} + T^s \Gamma_{sk}{}^i$$

- The covariant derivative of a (0,1) tensor

$$\nabla_k T_i \equiv \frac{\partial T_i}{\partial x^k} - T_s \Gamma_{ik}{}^s$$

- The covariant derivative of a (p,q) tensor

$$\nabla_k T_{j_1 j_2 \ldots j_q}{}^{i_1 i_2 \ldots i_p} \equiv \partial_k T_{j_1 j_2 \ldots j_q}{}^{i_1 i_2 \ldots i_p} + T_{j_1 j_2 \ldots j_q}{}^{s i_2 \ldots i_p} \Gamma_{sk}{}^i$$

$$+ T_{j_1 j_2 \ldots j_q}{}^{i_1 s \ldots i_p} \Gamma_{sk}{}^i + \cdots - T_{s j_2 \ldots j_q}{}^{i_1 i_2 \ldots i_p} \Gamma_{ik}{}^s - \cdots$$

Example 1: Geodesic Equation

Suppose r(t) is a curve parametrized by t, $r(t) = [x^1(t), x^2(t), \ldots, x^n(t)]$.
So the length of r from $t = a$ to $t = b$ is

$$\mathrm{S} = \int_a^b \sqrt{\left| g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right|}\, dt$$

We’re now going to find the equation of each coordinate that minimize L.
And that curve is called a geodesic.

$$S = \int_a^b \sqrt{\left| g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right|} dt = \int_a^b f(x^i, \dot{x}^i) dt = \int_a^b \sqrt{w} dt$$

By using Euler-Lagrange method: $\frac{\partial f}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}^k}\right) = 0$

$$\frac{\partial f}{\partial x^k} = \frac{1}{2\sqrt{w}} \frac{\partial g_{ij}}{\partial x^k} \frac{dx^i}{dt} \frac{dx^j}{dt}$$

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}^k}\right) = \frac{d}{dt}\left[\frac{1}{2\sqrt{w}}\left(g_{kj} \frac{dx^j}{dt} + g_{ik} \frac{dx^i}{dt}\right)\right] = \frac{d}{dt}\left(\frac{1}{\sqrt{w}} g_{kj} \frac{dx^j}{dt}\right)$$

$$\frac{\partial f}{\partial x^k} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}^i}\right) = \frac{1}{2\sqrt{w}}\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{dt}\frac{dx^j}{dt} - \frac{d}{dt}\left(\frac{1}{\sqrt{w}} g_{kj}\frac{dx^j}{dt}\right)$$

$$= \frac{1}{2\sqrt{w}}\left(\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{dt}\frac{dx^j}{dt} - 2\frac{\partial g_{kj}}{\partial x^s}\frac{\partial x^s}{\partial t}\frac{dx^j}{dt} - 2g_{kj}\frac{d^2x^j}{dt^2}\right) = 0$$

Note1, $\frac{d}{dt}\left(\frac{1}{\sqrt{w}} g_{kj}\frac{dx^j}{dt}\right) = \frac{1}{\sqrt{w}}\left(\frac{\partial g_{kj}}{\partial x^s}\frac{\partial x^s}{\partial t}\frac{dx^j}{dt} + g_{kj}\frac{d^2x^j}{dt^2}\right)$

Note2, $\frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{dt}\frac{dx^j}{dt} - 2\frac{\partial g_{kj}}{\partial x^s}\frac{\partial x^s}{\partial t}\frac{dx^j}{dt} = \frac{\partial g_{ij}}{\partial x^k}\frac{dx^i}{dt}\frac{dx^j}{dt} - \frac{\partial g_{ki}}{\partial x^j}\frac{dx^i}{dt}\frac{dx^j}{dt} - \frac{\partial g_{kj}}{\partial x^i}\frac{dx^i}{dt}\frac{dx^j}{dt}$

$$= -2\Gamma_{ijk}\frac{dx^i}{dt}\frac{dx^j}{dt}$$

$$\left[\frac{1}{2\sqrt{w}}\left(-2\Gamma_{ijk}\frac{dx^i}{dt}\frac{dx^j}{dt}-2g_{kj}\frac{d^2x^j}{dt^2}\right)=0\right]*g^{pk}$$

As we multiply g^{pk}, and knowing that $g^{pk}g_{kj}=\delta_j{}^p$; $g^{pk}\Gamma_{\ ijk}=\Gamma_{ij}{}^p$

Finally, we'll get the geodesic equation (for $p=1,2,\ldots,n$)

$$\Gamma_{ij}{}^p\frac{dx^i}{dt}\frac{dx^j}{dt}+\frac{d^2x^p}{dt^2}=0$$

It depicts a locally shortest path between two given points in a curved space, and it's useful in general relativity.

Example 2: Elasticity Theory

In classical elasticity theory, strain tensor can be written as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right) = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right)$$

Where u_i are the displacement vector components.

In general coordinates, we need to replace partial derivatives with covariant derivatives

$$\varepsilon_{ij} = \frac{1}{2}\left(\nabla_j u_i + \nabla_i u_j\right) = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j - \Gamma_{ij}{}^{k} u_k - \Gamma_{ji}{}^{k} u_k\right)$$

THE END