Introduction to General Stokes Theorem

物理系26級 林承諺

Introduction of Generalized Stokes theorem

The generalized Stokes theorem is a fundamental idea in math that links the integral of a special function, called a differential form, over a manifold to the integral over its boundary. It states,

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega$$

- Ω is an oriented n-dimensional manifold with boundary $\partial\Omega$,
- ω is a smooth $(n-1)$-form
- $d\omega$ is the exterior derivative of ω, resulting in an n-form

This rule generalizes simpler ideas like the fundamental theorem of calculus, Green's theorem, Divergence theorem, and the standard Stokes' theorem.

A brief introduction of p-form

In $\mathbb{R}^n$, a p-form can be expressed as:

$$\omega = \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \bigotimes dx^{\mu_2} \bigotimes \cdots \bigotimes dx^{\mu_p}$$

$$= \frac{1}{p!} \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}$$

(where $\wedge$ is the wedge product)

Some characteristics of wedge product (∧)

- Antisymmetry : $u \wedge v = -v \wedge u$
- Associativity : $(u \wedge v) \wedge w = u \wedge (v \wedge w)$
- Vanishes for Duplicates : $u \wedge u = 0$
- Relation to area and volume:

  ex. $dx \wedge dy \rightarrow dS$

  $dx \wedge dy \wedge dz \rightarrow dV$

  $dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{n-1} \rightarrow general\ volume$

Exterior derivative (can only act on p-form)

$$d: \omega \to d\omega$$

$$(\omega\ is\ a\ p - form, d\omega\ is\ a\ (p+1) form)$$

$$\omega = \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}$$

$$d\omega \stackrel{\text{def}}{=} \frac{1}{p!} (\frac{\partial}{\partial x^\nu} \omega_{\mu_1 \ldots \mu_p}) dx^\nu \wedge (dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p})$$

Applications in physics

In Classical Electromagnetism (n-dimensional Cartesian coordinate):

ex. Gauss's law

The Integral form: $\int_S \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{Q_{enc}}{\varepsilon_0} = \int_V \frac{\rho}{\varepsilon_0} dV$

By using divergence theorem, $\int_V \boldsymbol{\nabla} \cdot \boldsymbol{E}\, dV = \int_V \frac{\rho}{\varepsilon_0} dV$

Which gives the Differential form: $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$

Applications in physics

In Relativistic Electromagnetism(four-dimensional spacetime):

- In four-dimensional spacetime, the electromagnetic field is described by a 2-form F.
- The homogeneous Maxwell equations are $dF = 0$
- The inhomogeneous equations involve the current, written as $d\star F = J$ (Hodge star operator)

Consider a four-dimensional spacetime region M with boundary ∂M:

$$\int_M d\star F = \int_M J$$

Apply general stokes theorem:

$$\int_M J = \int_{\partial M} \star F$$

Here, $\int_M J$ represents the total charge passing through the four-dimensional region M, which can be interpreted as the integral of the current density over spacetime. The right-hand side, $\int_{\partial M} \star F$, is the integral of the dual field form over the boundary, which, in physical terms, relates to the electric flux through the boundary surfaces.