# *surface stress tensor*

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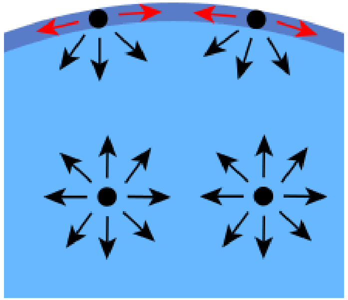
## *Classical Surface Tension*

Surface tension (scalar surface tension) is the "tensile force per unit length" acting on an interface, denoted by:

$$\gamma(N/m)$$

For example, a water droplet shrinks in air because surface tension tends to minimize the surface area of the interface.



Surface tension is commonly defined as a force per unit length acting along the interface.

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## *Why Do We Need a Stress Tensor?*

If the interface is curved (such as the surface of a water droplet), or if the surface tension varies with direction, then a single scalar γ is not sufficient to describe all the forces. In such cases, we introduce the **surface stress tensor:**

$$\sigma_s^{\alpha\beta}$$

This is a second-rank tensor that describes:The force in the $\alpha$ direction resulting from tension acting along the $\beta$ direction on the surface.

If the surface tension is isotropic, the tensor simplifies to:

$$\sigma_s^{\alpha\beta} = \gamma g^{\alpha\beta}$$

where $g^{\alpha\beta}$ is the surface metric tensor.

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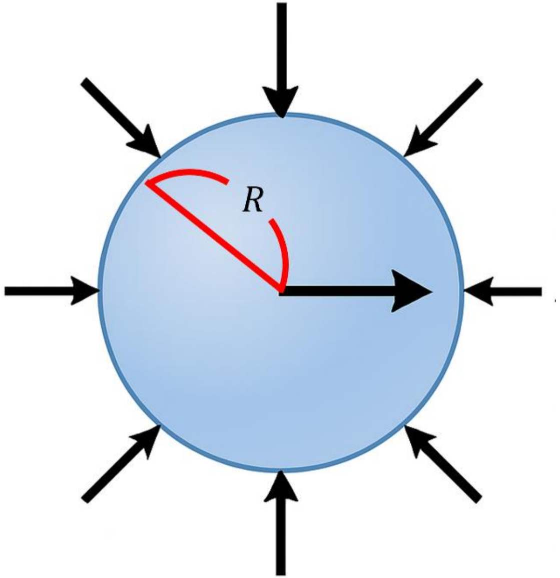
## *Spherical Droplet(Scalar Method)*

**Setup**:

Consider a spherical water droplet with radius $R$, where the external pressure is $P_{out}$, and the internal pressure is $P_{in}$.

The goal is to derive the relationship between the pressure difference $\Delta P$ , the surface tension $\gamma$, and the radius $R$.



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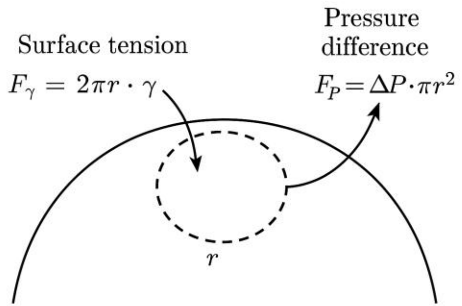
*Spherical Droplet(Scalar Method)*

Suppose we cut a small circular patch on the surface.




Because we want to analyze:
surface tension pulls inward along the boundary of the small circular patch (in the tangential direction)
the pressure difference pushes outward on the surface of the small circular patch (in the normal direction)
This leads us to the force balance equation:

$$F_\gamma = F_P$$

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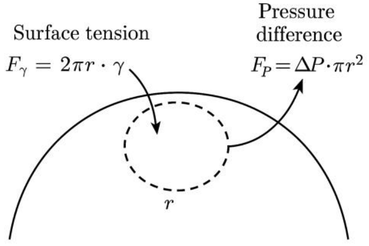
## *Spherical Droplet(Scalar Method)*

Set these two forces in equilibrium (static case):

$$F_\gamma = F_P = 2\pi r \cdot \gamma = \Delta P \cdot \pi r^2$$




So, we finally obtain:

$$\Delta P = \frac{2\gamma}{r}$$

If the small patch is part of a spherical surface, then the local radius of curvature is the sphere’s radius $R$:

$$\Delta P = \frac{2\gamma}{R}$$

This is the classical **Laplace–Young pressure difference formula**, derived from geometry and force balance.

## Laplace–Young Equation (General Curved Surface)

General Formula (Arbitrary Surface):

$$\Delta P = 2\gamma \frac{(k_1 + k_2)}{2} = 2\gamma(k_1 + k_2)$$

where $\mathrm{k}_1, \mathrm{k}_2$ are principal curvatures at the point (maximum and minimum normal curvatures) ,$\frac{(\mathrm{k}_1+\mathrm{k}_2)}{2}$ is the mean curvature.

Tensor Form of the Laplace–Young Equation:

$$\Delta P = \sigma_s^{\alpha\beta} K_{\alpha\beta}$$

where $\sigma_s^{\alpha\beta}$ is the surface stress tensor (isotropic or anisotropic), and $K_{\alpha\beta}$ is the second fundamental form (curvature tensor).

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## $Spherical\ Droplet(Tensor\ Method)$

From the contraction (inner product) between the surface stress tensor and the curvature tensor, we obtain:

$$\Delta P = \sigma_s^{\alpha\beta} K_{\alpha\beta}$$

For a spherical surface with isotropic surface tension:

$$\sigma_s^{\alpha\beta} = \gamma g^{\alpha\beta}, K_{\alpha\beta} = \frac{1}{R} g_{\alpha\beta}$$

Taking the contraction:

$$\Delta P = \gamma g^{\alpha\beta} \frac{1}{R} g_{\alpha\beta} = \frac{\gamma}{R} g^{\alpha\beta} g_{\alpha\beta}$$

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## $Spherical\ Droplet(Tensor\ Method)$

Since:

$$g^{\alpha\beta}g_{\alpha\beta} = \delta^{\alpha}_{\alpha} = 2$$

* While a water droplet is a three-dimensional body, the tensor fields under consideration—namely the surface stress tensor and the curvature tensor—are defined on its surface, which constitutes a two-dimensional manifold, $\delta^{\alpha}_{\alpha} = n(n - dimension)$

$$\sigma_s^{\alpha\beta} = \gamma g^{\alpha\beta}, K_{\alpha\beta} = \frac{1}{R}g_{\alpha\beta}$$

So, we finally obtain:

$$\Delta P = \frac{2\gamma}{r}$$

This matches the result from the scalar force-balance method, but the tensor-based approach has the advantage that it **can be generalized to any curved surface**.

# The End