Nother's theorem

Need to known: Lagrangian: $\mathcal{L}(\varphi, \partial_\mu \varphi, x)$, "motion" $S[\varphi] \equiv \int_M d^n x \, \mathcal{L}[\varphi(x), \partial_\mu \varphi(x), x]$.

Let $\mathbb{M}$ be an n-dim manifold & $\mathbb{T}$ a target manifold. Let $\mathbb{C}$ denote the configuration space consiting of smooth functions from $\mathbb{M} \to \mathbb{T}$

e.g. In classical mech.. $n=1$ $(t) \to \mathbb{T}$ general "space" $\to \{ x \to t, \ \varphi(t) = x(t), \ \underset{\substack{\downarrow \\ \text{only } t}}{\partial_\mu} \varphi(x) = \dot{x}(t) \}$

" field Theory. $n=4$ $(t, x_i) \to \mathbb{T}$ $\begin{cases} m \text{ scalar field } \varphi_1 \cdots \varphi_m \to \mathbb{R}^m \ (\text{real}) \\ 1 \text{ vector field } \to \mathbb{R}^3 \ (\text{real}) \end{cases}$

Let a functional $S: \mathbb{C} \to \mathbb{R}$:

If given a boundary condition than ① $\dfrac{\delta}{\delta \varphi(x)} S[\phi] = 0$

② : φ satisfy the boundary condition

Now, Let a infinitesimal transformation Q on $\mathbf{C}$ satisfy $Q\left[\int_N d^n x\, \mathcal{L}\right] = \int_{\partial N} ds_\mu\, f^\mu[\phi(x), \partial\phi, \partial\partial\phi, \cdots]$

for all sub manifold N

$\xRightarrow{\text{diverge theorem}}$ $Q[\mathcal{L}(x)] = \partial_\mu f^\mu(x)$ for all x.

$(\mathcal{L}(x) = \mathcal{L}[\varphi(x), \partial_\mu \varphi(x), x])$

Q is the gererator of a 1-parameter symmetry Lie group

How to gererate:

$\varphi \to \varphi' = \varphi + \delta\varphi, \quad \delta\varphi = \lim_{\varepsilon_r \to 0} \sum_r \varepsilon_r Q_r$

$Q \to \Psi \quad \varphi \mapsto \varphi + \varepsilon\Psi$

$\Rightarrow \mathcal{L} \mapsto \mathcal{L} + \varepsilon\, \partial_\mu \Lambda^\mu$

$$\Rightarrow Q\left[\int_N d^n x \mathcal{L}\right] = \int_N d^n x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right] Q[\phi] + \int_{\partial N} ds_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} Q[\phi]$$

"surfaces" on the manifold

$$= \int_{\partial N} ds_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} Q[\phi]$$

some vector fcn

$$\Rightarrow \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} Q[\phi] - f^\mu\right] = 0 \qquad J^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} Q[\phi] - f^\mu$$

$\to$ "Noether current"

$\Rightarrow \partial_\mu J^\mu = 0$ (continuity eqn) $\underbrace{\frac{\partial \rho}{\partial t}}_{\partial_0 J^0} + \nabla \cdot (\underbrace{\rho \vec{v}}_{J^i}) = 0 \;\; (\to \partial_i J^i)$ $\longrightarrow$ "Noether charge" conserves.

$\hookrightarrow \int_\Sigma j^\mu ds^\mu = \int j^0 d^3x \quad \Sigma$: ($t$=const)'s space super surface $\to ds^\mu = \begin{pmatrix} d^3x \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\rho = J^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} Q[\phi] - f^0$$

Example ①: U(1) phase symmetry → charge conserve

a complex scalar field $\phi(x)$

$$\mathcal{L} = \underbrace{\partial_\mu \phi^* \partial^\mu \phi}_{\text{kinetic term}} - \underbrace{\overset{\substack{\text{mass parameter}\\ \downarrow}}{m^2} \phi^* \phi}_{\text{potential term}}$$

U(1) symmetry transform: $\phi \to e^{i\alpha}\phi \Rightarrow \delta\phi = i\alpha\phi$

$\phi^* \to e^{-i\alpha}\phi^* \qquad \delta\phi^* = -i\alpha\phi^*$

$\because \delta\mathcal{L} = 0 \therefore F^\mu = 0$

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^* = \partial^\mu\phi^* \, i\alpha\phi + (-i\alpha\phi^*)\,\partial^\mu\phi = i\cancel{\alpha}\left(\partial^\mu\phi^*\cdot\phi - \phi^*\cdot\partial^\mu\phi\right)$$

α infinitesimal parameter

$$\to J^\mu = i\left(\partial^\mu\phi^*\cdot\phi - \phi^*\cdot\partial^\mu\phi\right)$$

$$Q = \int d x^3 j^0 = i\int d^3x \left(\partial^0\phi^*\cdot\phi - \phi^*\cdot\partial^0\phi\right) \qquad \frac{dQ}{dt} = 0$$

Q is the "charge" $\langle Q\rangle$, "charge operator" $\hat{Q}$, "generator" $Q[\phi]$

E.g ②

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}$$

$T^{\mu\nu}$: Noether current

$$\partial_\mu T^{\mu\nu} = 0$$

$E = \int d^3x\, T^{00}$ (total) Energy conserve $\frac{dE}{dt} = 0$

$\Rightarrow$

$P^i = \int d^3x\, T^{0i}$ (total) Momentum conserve $\frac{dP^i}{dt} = 0$

E.g ③ QM:

generator: $\hat{P}^\mu$ $\quad P^0 = \hat{H}$: Hamiltonian

P^i: momentum operator

$(\hbar = 1)$

$$\phi(x^\mu) \to e^{i\alpha^\mu\hat{P}_\mu}\phi(x^\mu)e^{-i\alpha^\mu\hat{P}_\mu} = \phi(x^\mu + \alpha^\mu)$$

if $[\hat{H},\phi] = i\partial_t\phi$, than $\hat{H}$ is the generator of time translation → energy conserve

if $[P^i,\phi] = -i\partial_t\phi$, than P^i is the generator of space translation → momentum conserve

$$\frac{d}{dt}\langle\phi|\hat{P}^\mu|\phi\rangle = 0$$