HYPERGEOMETRIC FUNCTIONS

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What is hypergeometric function ?

- We can see a ODE:

$$x(1-x)y''(x) + [c-(a+b+1)x]y'(x) - ab\ y(x) = 0$$

The gereral solusion of this ODE is:

$$y(x) = {}_2F_1(a,b;c;x)$$

$$= 1 + \frac{a\,b}{c}\frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{x^2}{2!} + \cdots, \quad c \neq 0, -1, -2, -3, \ldots,$$

We can define $(a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad (a)_0 = 1.$

Using this notation, the hypergeometric function becomes:

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}.$$

Confluent hypergeometric function

- We can see a ODE:

$$xy'' + (c - x)y' - ay = 0$$

The gereral solusion of this ODE is:

$$y_1(x) = 1 + \frac{a}{c}\frac{x}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \cdots \equiv M(a, c; x),$$

$$y_2(x) = x^{1-c} M(a - c + 1, 2 - c; x),$$

We can define $M(a, c; x) =_1 F_1(a; c; x)$

Using this notation, the hypergeometric function becomes:

$${}_1F_1(a; c; x) = \Sigma_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} z^n$$

Confluent hypergeometric function

- The relation to hypergeometric function

$$ {}_1F_1(a;c;x) = \lim_{b\to\infty} \left({}_2F_1(a,b;c;x)\right) $$

- The singularity of hypergeometric function
- ${}_2F_1$ have three regular singularity $z = 0, z = 1, z \to \infty$
- ${}_1F_1$ have one regular singularity $z = 0$

  one irregular singularity $z \to \infty$
- The confluent hypergeometric function arises when two regular singularities of the Gauss hypergeometric equation merge into a single irregular singularity at infinity.
- This merging of singularities is not just mathematical — it reflects the change in behavior of physical systems, for example, from oscillatory to exponential decay near infinity.

Relation to Other Special Functions

- There is some example:

$$F(a,b,b;x) = (1-x)^{-a},$$

$$F(1,1,2;-x) = x^{-1}\ln(1+x),$$

$$\lim_{m\to\infty} F(1,m,1;x/m) = e^x,$$

$$F(\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2};\sin^2 x) = \cos x,$$

$$F(\tfrac{1}{2},p,p;\sin^2 x) = \sec x,$$

$$F(\tfrac{1}{2},\tfrac{1}{2},\tfrac{3}{2};x^2) = x^{-1}\sin^{-1} x,$$

$$F(\tfrac{1}{2},1,\tfrac{3}{2};-x^2) = x^{-1}\tan^{-1} x,$$

$$F(\tfrac{1}{2},1,\tfrac{3}{2};x^2) = \tfrac{1}{2}x^{-1}\ln[(1+x)/(1-x)],$$

$$F(m+1,-m,1;(1-x)/2) = P_m(x),$$

$$F(m,-m,\tfrac{1}{2};(1-x)/2) = T_m(x),$$

Legendre polynomial

- Legendre differential equation:

$$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$$

- hypergeometric function:

$$x(1-x)y''(x) + [c-(a+b+1)x]y'(x) - ab\,y(x) = 0$$

- Change variable: $z = \frac{1-x}{2}$
- By this change variable, we can Transforms into the hypergeometric equation

Another special function

$$P_\ell(x) = {}_2F_1\left(-\ell, \ell+1; 1; \frac{1-x}{2}\right) ;$$ Legendre

$$P_\ell^m(x) = \frac{(\ell+m)!}{(\ell-m)!}\frac{(1-x^2)^{m/2}}{2^m m!}{}_2F_1\left(m-\ell, m+\ell+1; m+1; \frac{1-x}{2}\right) ;$$ Associated Legendre

$$J_n(x) = \frac{e^{-ix}}{n!}\left(\frac{x}{2}\right)^n {}_1F_1(n+\tfrac{1}{2}; 2n+1; 2ix) ;$$ Bessel

$$H_{2n}(x) = (-1)^n\frac{(2n)!}{n!}{}_1F_1(-n; \tfrac{1}{2}; x^2) ;$$

Hermite

$$H_{2n+1}(x) = (-1)^n\frac{2(2n+1)!}{n!}x\,{}_1F_1(-n; \tfrac{3}{2}; x^2) ;$$

$$L_n(x) = {}_1F_1(-n; 1; x) ;$$ Laguerre

$$L_n^k(x) = \frac{\Gamma(n+k+1)}{n!\Gamma(k+1)}{}_1F_1(-n; k+1; x) ;$$ Associated Laguerre

$$T_n(x) = {}_2F_1\left(-n, n; \tfrac{1}{2}; \frac{1-x}{2}\right) .$$ Chebyshev

Application I: Potentials(Spherical Coordinates)

- Laplace's equation(assume azimuthal symmetry):

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0.$$

- We use separation of variables:

$$V(r,\theta)=R(r)\Theta(\theta).$$

- The solution is:

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta).$$

Application II: Harmonic oscillator

- Time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi.$$

- The solution is:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi)\, e^{-\xi^2/2}.$$