Cemtral Limit Theorem

Jen-Jie Hung

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Description of Central Limit Theorem

Theorem

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables,each having mean μ and variance σ^2. Then the distribution of

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as $n \to \infty$. That is, for $-\infty < a < \infty$,

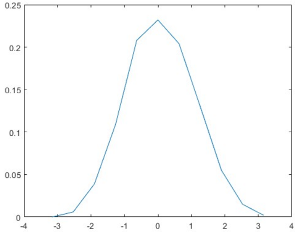
$$P\left\{\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leqslant a\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx \text{ as } n \to \infty$$

Example

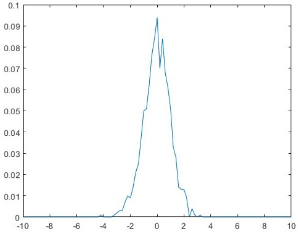
Let $X_1, X_2, ..., X_n$ be Bernoulli random variables with $P\{X_i = 1\} = p, i = 1, 2, ..., n$,then the mean $\mu = np$, the variance $\sigma^2 = pq$.By Central Limit Theorem,

$$\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

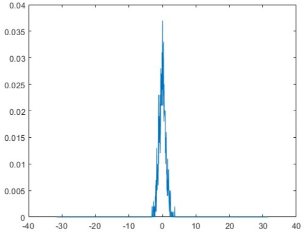
tends to the standard normal as n $\rightarrow \infty$.



(a) p=0.5,n=10

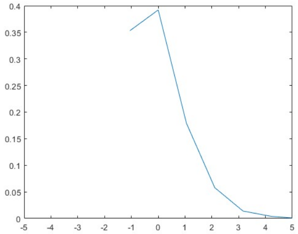


(b) p=0.5,n=100

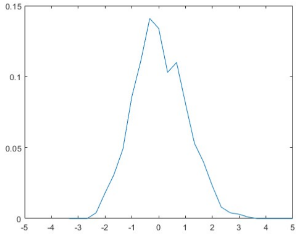


(c) p=0.5,n=1000

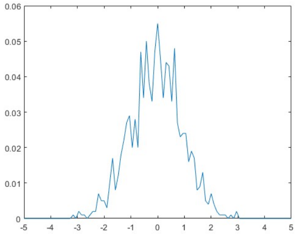
Example



(a) p=0.1,n=10



(b) p=0.1,n=100



(c) p=0.1,n=1000

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Moment generating function

Definition

The moment generating function M(t) of the random variable X is defined for all real values of t by

$$M(t) = \begin{cases} \sum_x e^{tx} p(x) & \text{if } X \text{ is discrete with mass function } p(x) \\ \int_{-\infty}^{\infty} e^{tx} f(x)dx & \text{if } X \text{ is continuous with density } f(x) \end{cases} \tag{1}$$

Example

Let X be a Bernoulli variable with $P\{X = 1\} = p$, then

$$M(t) = e^{0\times t}(1-p) + e^{1\times t}p = pe^t + 1 - p \tag{2}$$

Lemma

Lemma

Let $Z_1, Z_2, \ldots$ be a sequence of random variables having distribution functions F_{Z_n} and moment generating functions $M_{Z_n}, n \geqslant 1$, and let Z be a random variable having distribution function F_Z and moment generating function M_Z. If $M_{Z_n}(t) \to M_Z(t)$ for all t, then $F_{Z_n}(t) \to F_Z(t)$ for all t at which $F_Z(t)$ is continuous.

Remark

Moment generating function of normal random variable

$$M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}} \tag{3}$$

Proof

Proof.

Let $X_1, X_2, ..., X_n$ be Bernoulli random variables with $P\{X_i = 1\} = p, i = 1, 2, ..., n$,then the mean $\mu = p$, the variance $\sigma^2 = pq$.

$$\begin{aligned} M_{\frac{X_1+\cdots+X_n-n\mu}{\sigma\sqrt{n}}}(t) &= \sum_{k=0}^{n} e^{\frac{(k-n\mu)t}{\sqrt{np(1-p)}}} C_k^n p^k (1-p)^{n-k} \\ &= e^{\frac{-npt}{\sqrt{np(1-p)}}} \sum_{k=0}^{n} C_k^n (pe^{\frac{t}{\sqrt{np(1-p)}}})^k (1-p)^{n-k} \\ &= e^{\frac{-npt}{\sqrt{np(1-p)}}} (pe^{\frac{t}{\sqrt{np(1-p)}}} + (1-p))^n \end{aligned} \tag{4}$$

□

Proof

Proof.

By Taylor expansion $e^{\frac{t}{\sqrt{np(1-p)}}} \approx 1 + \frac{t}{\sqrt{np(1-p)}} + \frac{t^2}{2np(1-p)}$ if n large enough, so

$$
\begin{aligned}
M_{\frac{X_1+\cdots+X_n-n\mu}{\sigma\sqrt{n}}}(t) &\approx e^{\frac{-npt}{\sqrt{np(1-p)}}}\left(p\left(1+\frac{t}{\sqrt{np(1-p)}}+\frac{t^2}{2np(1-p)}\right)+(1-p)\right)^n \\
&= e^{\frac{-npt}{\sqrt{np(1-p)}}}\left(1+\frac{\frac{pt\sqrt{n}}{\sqrt{p(1-p)}}+\frac{pt^2}{2p(1-p))}}{n}\right)^n \\
&\approx e^{\frac{-npt}{\sqrt{np(1-p)}}}\, e^{\frac{pt\sqrt{n}}{\sqrt{p(1-p)}}+\frac{t^2}{2(1-p))}} = e^{\frac{t^2}{2(1-p)}}
\end{aligned} \tag{5}
$$

$\square$

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Application

1. Noise in physic system
2. Survey

Reference

Sheldon Ross, *A First Course in Probability*