Special Discrete Distributions

From Binomial Distribution to Poisson Distribution

Experiment

success

failure

Bernoulli trial

Bernoulli trial

p

$1 - p \equiv q$

$$p(x) = \begin{cases} q, & x = 0 \\ p, & x = 1 \\ 0, & other \end{cases}$$

Binomial random variable

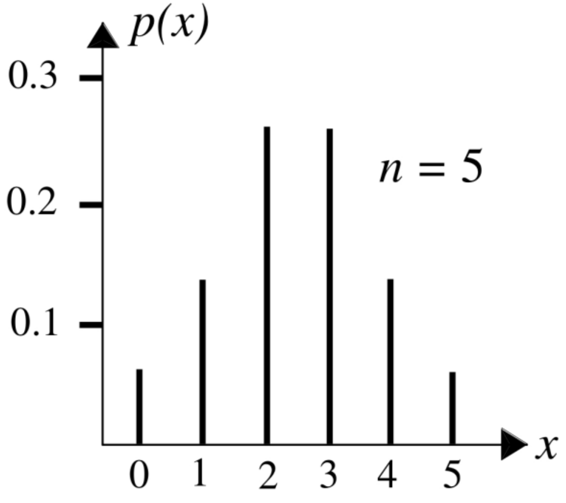
$\times n$

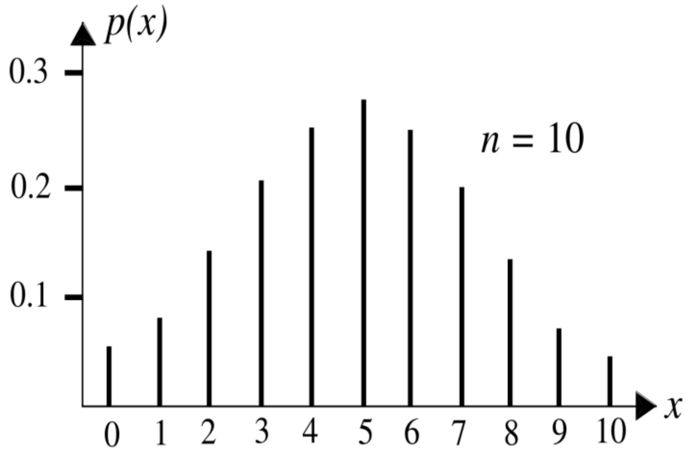
All p(success) independently

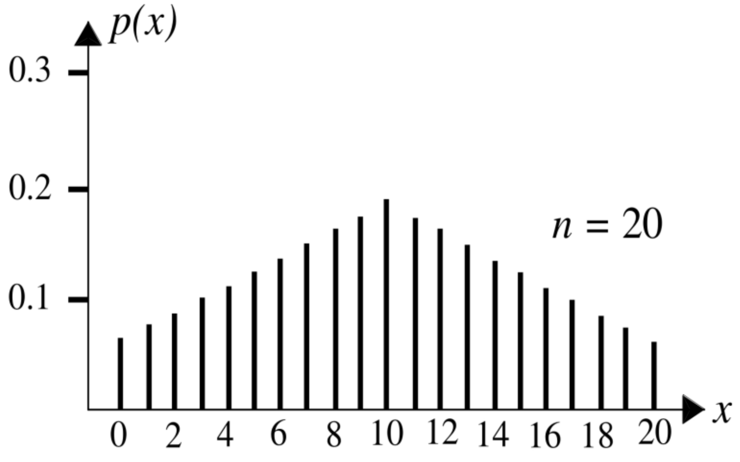
$$p(x) = P(X = x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & if\ x = 0, 1, 2, \dots, n \\ 0, & elsewhere \end{cases}$$

The number of successes
(A binomial random variable)

Distribution







(n, p) $(5, 0.5)$ $(10, 0.5)$ $(20, 0.5)$

Small probability

$$n \to \infty \text{、} p \to 0 \text{、} np = \lambda(moderate)$$

Large number of trials

Poisson random variable

$$P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \frac{n!}{(n-i)!\,i!}\left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{n(n-1)\cdots(n-i+1)}{n^i}\frac{\lambda^i}{i!}\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}$$

$$\to \frac{e^{-\lambda}\lambda^i}{i!}$$

- $np = \lambda$
- $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^i = 1$
- $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$
- $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$

Good

- $p < 0.1 \; and \; np < 10$

- If $np > 10 => normal \; distribution$

Example: the number of incoming calls

4 calls per hour

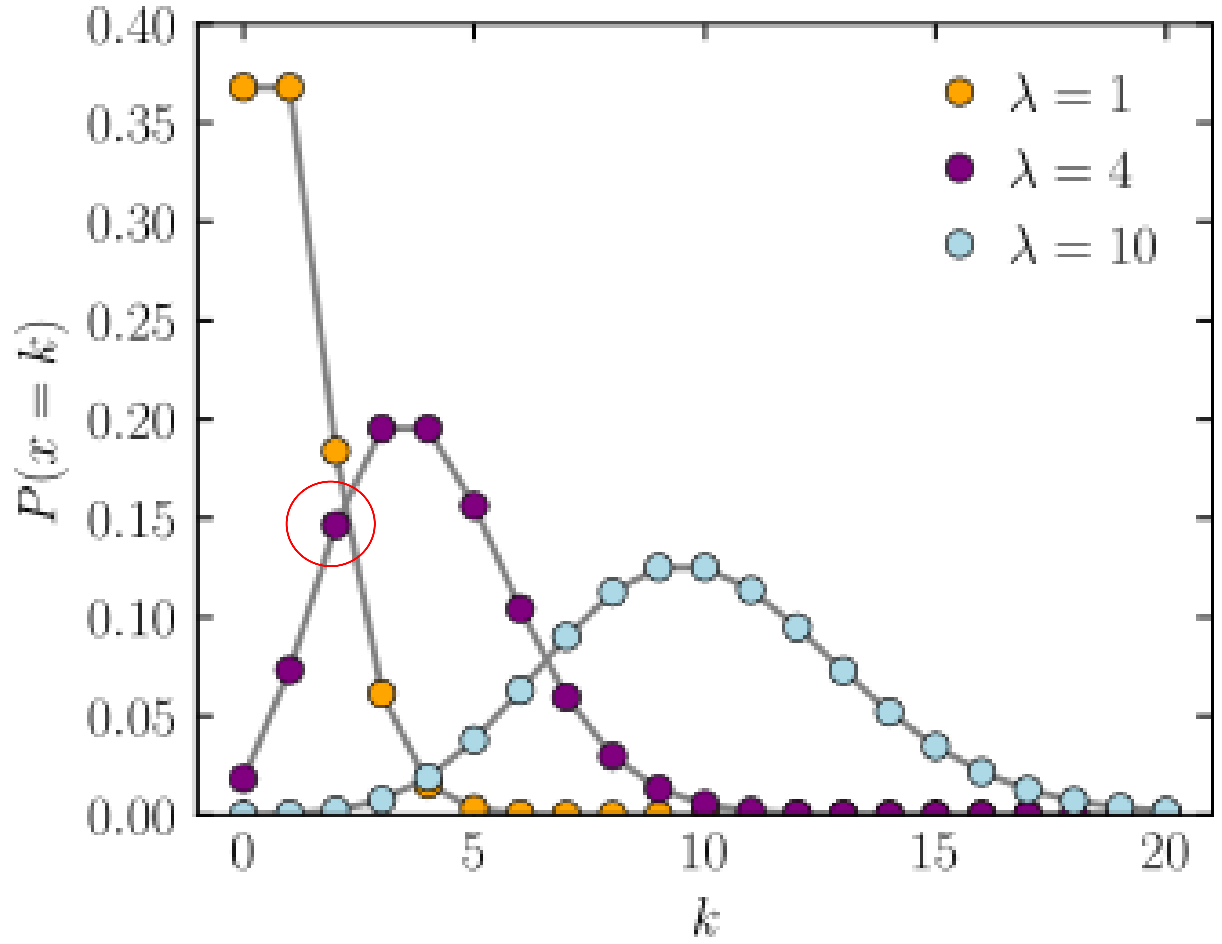
What is the probability of receiving 2 calls in the next hour?

$\lambda = 4$

$$P(X = i) = \frac{e^{-\lambda}\lambda^{i}}{i!}$$

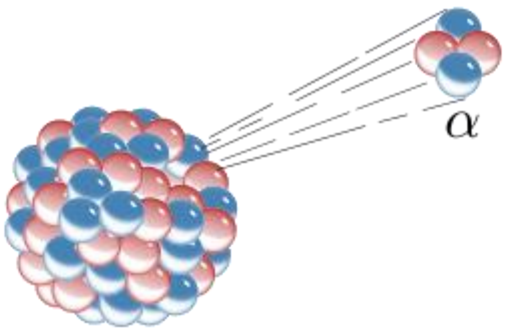
$$P(X = 2) = \frac{e^{-4}4^{2}}{2!} \approx 0.1465$$

Ans: 14.65%



0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
$P(x = k)$
0
5
10
15
20
k
$\lambda = 1$
$\lambda = 4$
$\lambda = 10$

Example: the radioactive decay of elements



Every gram of this element, on average
Emits 3.9 α particle per second

What is the probability that during the next second the number of alpha particles emitted from 1 gram is

(a) At most 6;
(b) At least 2;
(c) At least 3 and at most 6?

Solution

Large n   Emit an alpha particle during the next second(success)

X, the number of alpha particle emitted during the next second

$E(X) = 3.9, np = 3.9, p = 3.9/n$

$$\lambda = 3.9 \Rightarrow P(X = n) = \frac{e^{-3.9}(3.9)^n}{n!}$$

Solution

$$(a)\ P(X \leq 6) = \sum_{n=0}^{6} \frac{e^{-3.9}(3.9)^n}{n!} \approx 0.899$$

$$(b)\ P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 0.901$$

$$(c)\ P(3 \leq X \leq 6) = \sum_{n=3}^{6} \frac{e^{-3.9}(3.9)^n}{n!} \approx 0.646$$

Poisson process

1. **Stationarity:** The average rate of occurrence is **constant** over time.
2. **Independent Increment:** Each event occurs **independently** of the others.
3. **Orderliness:** Two events cannot occur at exactly the same instant.

Appendix

	$E(X)$	$Var(X)$	σ_x
Binomial	np	$np(1-p)$	$\sqrt{np(1-p)}$
Poisson	λ	λ	$\sqrt{\lambda}$

Thank you for listening