Large Deviation Theory

Table of Content

Intro to Large Deviation Principle

What is it?

- a mathematical framework for quantifying the probabilities of rare events in systems. .

Intro to Large Deviation Principle

- **Problem definition**
- A set of random variable X_i with a certain probability distribution $P(X_i \in\ A) = \int_A p(x)dx$ for A is an interval
- They are **independent and identically distributed**, so we have:

$$P(X_1 = x_1 \wedge X_2 = x_2 \wedge \ldots X_N = x_N) = \prod_{i=1}^{N} p(x_i)dx_i$$

- define:

$$S_N\ = \frac{1}{N}\sum_{i=1}^{N} X\_i$$

Large Deviation Principle

- The principle gives

$$\lim_{N\to\infty} -\frac{1}{N} P(S_n = s) = I(s)$$

$$\Rightarrow P(S_n = s) = \lim_{N\to\infty} e^{-NI(s)}$$

- Where I(s) is the rate function. Show the probability of a large enough system to has the certain statistical value.

Contraction Principle

- A set random variable $\{X_i\}$ satisfy the Large Deviation Principle with rate function $I(x)$
- There is a set of new variable $\{Y_i = f(X_i)\}$ (f is a continuous function), it will satisfy Large Deviation Principle as well and its rate function is

$$J(y) = \inf_{x \in f^{-1}(y)} I(x)$$

- inf means the infimum(the greatest value for all x) of I(x) over all the x that maps to y.

Contraction Principle(Application)

- With this principle, we can use the Large Deviation Principle to almost all statistical quantities, such as energy, magnetization…

- The property of microstate(probability of certain states)

  $\Rightarrow$ Physical quantities in macro

Example of Large Deviation Principle-Entropy Density

Consider a N-scale Ising Model:

- each spin has state $\sigma_i \in \{-1, 1\}$, and our configuration space $\Omega_N = \{\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)\}$
- The probability of a certain state is

$$P(\sigma) = \frac{1}{|\Omega_N|} = \frac{1}{|\Omega|^N} = \frac{1}{2^N}$$

The probability of this Ising model to has the energy per spin within an interval E, $h_N(\sigma) \in E = [\epsilon, \epsilon + d\epsilon]$, is

$$P(h_N(\sigma) \in E) = \frac{\Omega(h_N \in E)}{|\Omega_N|}$$

Example of Large Deviation Principle- Entropy Density

- And the rate function of this gives

$$I(\epsilon) = \lim_{N\to\infty} -\frac{1}{N} lnP(h_N(\sigma) \in E)$$

$$= \lim_{N\to\infty} \frac{\ln|\Omega_N|}{N} - \frac{\ln\Omega(h_N \in \ E)}{N}$$ = ln(2) −s(ϵ)

- $s(\epsilon)$ is exactly the average entropy of every spin (or "entropy density") at energy ϵ
- The rate function(determine the decay of prob at certain energy) is given by the "difference between the entropy of all case and the entropy of certain situation"

Other Application of Large Deviation Theory

- **Equilibrium Statistical Mechanics**
  - **Equivalence of Ensembles**: LDT formalizes when microcanonical and canonical ensembles yield the same macroscopic predictions. The rate function (from LDT) links the entropy and the Legendre transform of the free energy.
  - **Thermodynamic Potentials**: LDT shows how thermodynamic quantities like entropy and free energy emerge as rate functions describing fluctuations.
  - **Fluctuation:** It can work with the law of large numbers and central limit theorem by providing exponential estimates of unlikely fluctuations
- There are more to analyze in: Non-equilibrium Systems, Quantum Systems and Biophysics

Reference

- A basic introduction to large deviations: Theory, applications, simulations | Hugo Touchette
- The large deviation approach to statistical mechanics | Hugo Touchette