Spontaneous Symmetry Breaking and Coset Spaces

An introduction to symmetry, vacuum structure, and Goldstone theorem

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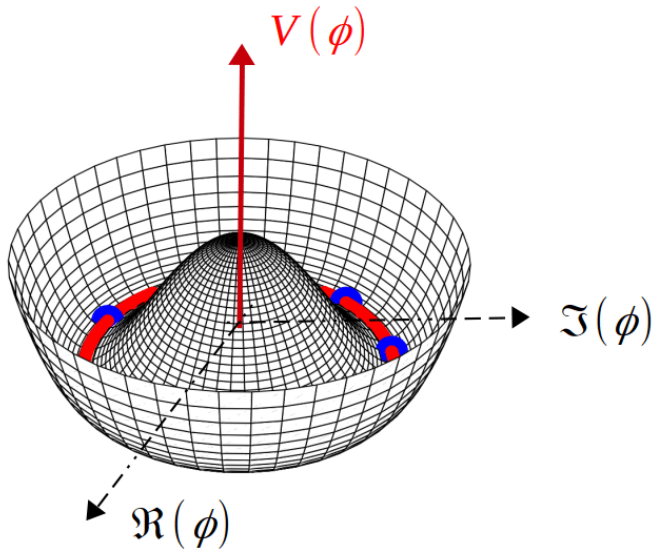
What is Spontaneous Symmetry Breaking?

- Laws have symmetry → The Lagrangian $\mathcal{L}$ is invariant under a group G
- Vacuum does not → The ground state only invariant under $H \subset G$



Vacuum state is a lowest state in field theory, just like the ground state in quantum state

Coset spaces

- $G/H = \{\, gH \mid g \in G\}$
- $G = \{\, I, a, b, ab, a^2, a^2b\}, a^3 = b^2 = I, ba = a^2b$
- Suppose $H = \{I, b\}$ → subgroup
- Then $aH = \{\, a, ab\}$ → (left coset)

$*$	$\boldsymbol{I}$	$\boldsymbol{a}$	$\boldsymbol{a^2}$	$\boldsymbol{b}$	$\boldsymbol{ab}$	$\boldsymbol{a^2b}$
$\boldsymbol{I}$	I	a	a^2	b	ab	a^2b
$\boldsymbol{a}$	a	a^2	I	ab	a^2b	b
$\boldsymbol{a^2}$	a^2	I	a	a^2b	b	ab
$\boldsymbol{b}$	b	a^2b	ab	I	a^2	a
$\boldsymbol{ab}$	ab	b	a^2b	a	I	a^2
$\boldsymbol{a^2b}$	a^2b	ab	b	a^2	a	I

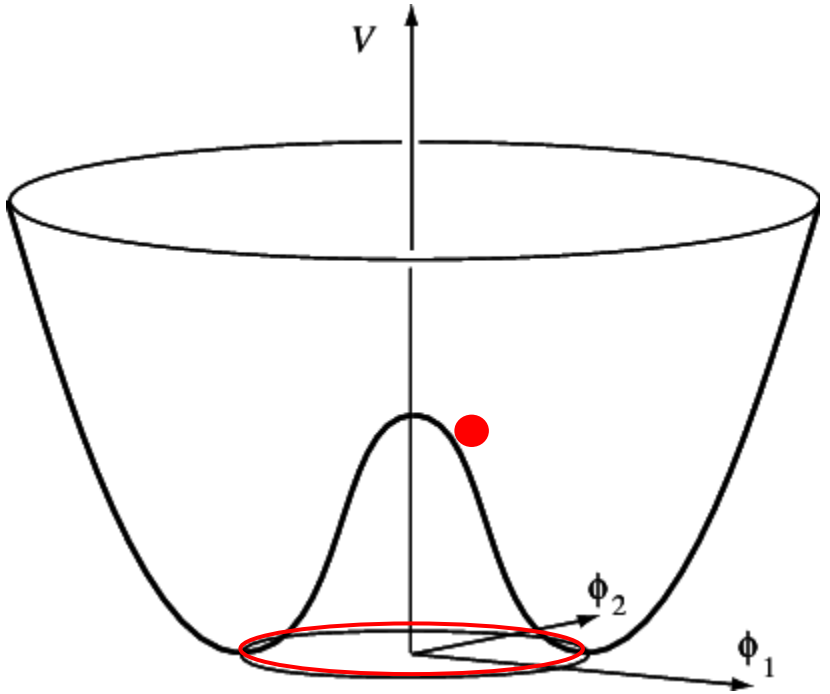
Goldstone theorem

- Let us consider a physical theory at quantum level, respectively at the classical level, with a global continuous symmetry group G such that it is spontaneously broken to a subgroup H different from G and that the notion of gap is well defined. Then, the spectrum of the theory will contain at least one gapless particle, respectively at least one gapless mode.

Goldstone Theorem

- $\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - V(\phi)$
- $V(\phi\ ) = \lambda(|\phi|^2 - v^2)^2$
- Vacuum state: $|\phi|^2 = v^2$
- $G = U(1)\ \left(\phi \to e^{i\alpha}\phi, g = e^{i\alpha}\right)$
- $H = \{identity\}$
- $G/H = U(1)\ /\ \{e\} \cong S^1 \cong U(1)$

$$\dim(G) - \dim(H)$$



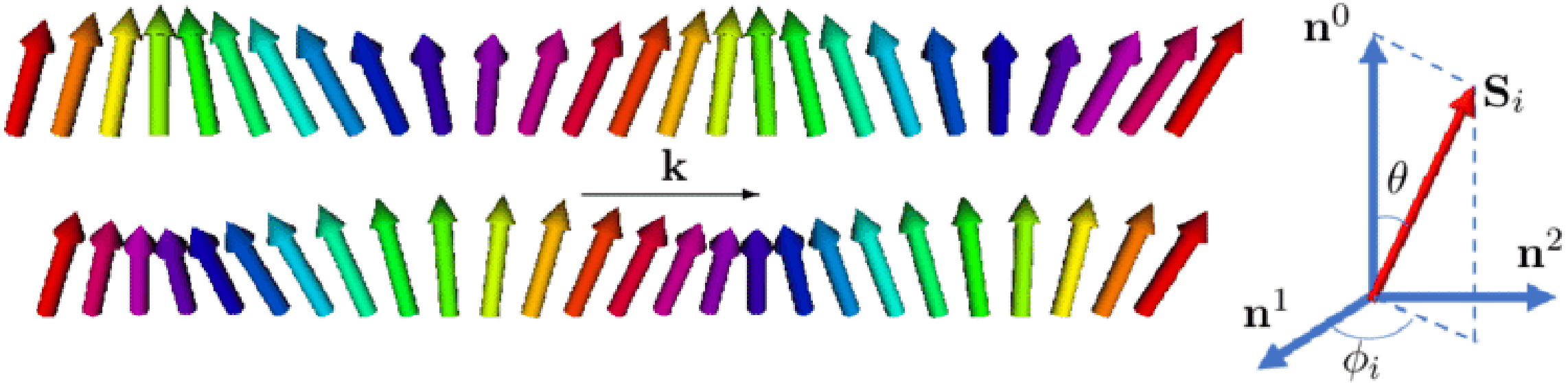
Coset space

Ferromagnet

- $SO(3) \rightarrow SO(2)$
- Consider magnetic moment is rotation symmetry in high temperature ($SO(3)$: Three direction of rotation)
- After below the T_c critical temperature, breaking into rotating in one direction ($SO(2)$)
- $\dim\big(SO(3)\big) - \dim\big(SO(2)\big) = 3 - 1 = 2$
- $G/H = SO(3)/SO(2) \cong S^2(coset\ construction)$

Coset Degree of Freedom(S^2)

- $\pi(x) = \big(\theta(x), \phi(x)\big) \rightarrow magnon\ (Goldstone\ mode)$
- Generate a spin wave



Summary

- Spontaneous Symmetry Breaking: $G \to H$
- Coset spaces G/H classify broken directions
- Goldstone theorem: broken symmetry → massless modes