Casimir operators & Particles

1. Quantum symmetry

Physical (continuous) symmetry $\longrightarrow$ Quantum mechanics

$\downarrow$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\downarrow$

Lie groups $\xrightarrow{\text{Lie group representation}}$ Hilbert spaces

$\downarrow$ infinitesmal $\qquad$ Lie alg. rep.

Lie algebra $\nearrow$

Physical quantities of a particle

(v.s. no-hair thm. of black holes)

- mass $m \leftarrow \mathbb{R}^{1,3}$ (spacetime trans.)
- spin $s \longleftarrow SU(2)$ $\left(\begin{array}{c} SU(2)\oplus SU(2) \\ \shortparallel \\ SO(1,3)\end{array}\right)$
- charge $q \longleftarrow U(1)$

Casimir operator $(\mathfrak{g}, [,])$ w/ scalar product $K(T_a, T_b) = K_{ab}$ $\leftarrow$ Killing form if $\mathfrak{g}$ is nice enough (e.g. semisimple)

$\hookrightarrow \{T_a\}$ basis of $\mathfrak{g}$

Def. $\rho : \mathfrak{g} \longrightarrow \mathcal{L}(V)$ rep. of Lie algebra $\mathfrak{g}$. Casimir operator = rep. of scalar product i.e.

$$\hat{C}_2 = \rho(T_a)\rho(T^a) = K^{ab}\rho(T_a)\rho(T_b).$$

Rmk. $\hat{C}_2$ is indep. of the choice of basis $\Rightarrow$ We can always chose orthonormal basis

$$K(\underbrace{T_a, T_b}_{\text{orthonormal basis}}) = \pm\delta_{ab}$$

$\pm$: possible indefinite e.g. Minkowski metric

Prop. $\hat{C}_2$ commutes with all rep. operators, i.e. $[C, \rho(X)] = 0 \;\forall\, X \in \mathfrak{g}$.

p.f. $\hat{C}_2\rho(T_c) = K^{ab}\rho(T_a)\rho(T_b)\rho(T_c) = K^{ab}\rho(T_a)K_{bc}K^{bc}\rho(T_b)\rho(T_c) = K^{ab}K_{bc}\rho(T_a)\hat{C}_2 = \delta^a_c\rho(T_a)\hat{C}_2 = \rho(T_c)\hat{C}_2$ $\square$

$\Rightarrow$ If ρ is irred., then $\hat{C}_2 = \mu\mathbb{1}$! (By Schur's lemma in rep. theory)

2. SU(2) representation

Pauli matrices $\sigma_1, \sigma_2, \sigma_3$, $J_i = \frac{1}{2}\sigma_i \Rightarrow [J_i, J_j] = i\,\varepsilon_{ijk}J_k$.

why choose them?

One considers $J^2 := J_1^2 + J_2^2 + J_3^2$ $\longrightarrow$ $\{\sigma_i\}$ (or $\{J_i\}$) forms an orthogonal basis of "Killing form" on $\mathfrak{su}(2)$

In QM

$J^2|j,m\rangle = j(j+1)|j,m\rangle$ w/ $j \in \frac{1}{2}\mathbb{Z}$, why?

Def. (Formal definition) The Killing form on $\mathfrak{g}$ is defined by $K(X,Y) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_X \mathrm{ad}_Y)$ for $X, Y \in \mathfrak{g}$

$(\mathrm{ad}_X Y \hat{=} [X,Y]$ in $\mathfrak{g})$.

For $\mathfrak{su}(n)$: $K(X,Y) = 2n\,\mathrm{tr}_{\mathbb{C}^n}(XY)$ $(n \geq 2)$ $\Rightarrow$ Killing form on $\mathfrak{su}(2)$ is $K(X,Y) = 4\,\mathrm{tr}_{\mathbb{C}^2}(XY)$

$\{J_1, J_2, J_3\}$ forms orthonormal basis

$\Rightarrow J^2$ is the Casimir operator of $\mathfrak{su}(2)$ $\Rightarrow$ commutes with J_i, $J^2|v\rangle = \lambda|v\rangle \ \forall |v\rangle \in \mathcal{H}$.

simultaneously diagonalizable

$\Rightarrow |\lambda, m\rangle$ common eigenstate of J^2 and J_3 $\begin{cases} J^2|\lambda,m\rangle = \lambda|\lambda,m\rangle \\ J_3|\lambda,m\rangle = m|\lambda,m\rangle \end{cases}$, $\underline{J_\pm|\lambda,m\rangle = C_\pm(\lambda,m)|\lambda,m\pm1\rangle}$

$J_\pm = J_1 \pm J_2$ (By $[J_i, J_j] = i\varepsilon_{ijk}J_k$)

(positive definite)

$J^2 - J_3^2 = J_1^2 + J_2^2 \geq 0 \Rightarrow \lambda - m^2 \geq 0$. $\Rightarrow \begin{cases} 0 = J_-J_+|\lambda,j\rangle = (\lambda - j^2 - j)|\lambda,j\rangle = 0, & j = \max \text{ of } \{m\} \\ 0 = J_+J_-|\lambda,k\rangle = (\lambda - k^2 + k)|\lambda,k\rangle = 0, & k = \min \text{ of } \{m\} \end{cases}$

(ex. why $k = j+1$ is wrong)

$\Rightarrow j(j+1) = k(k-1) = \lambda \Rightarrow j = -k \Rightarrow j - k = 2j \in \mathbb{Z} \Rightarrow \boxed{j \in \frac{1}{2}\mathbb{Z},\ \lambda = j(j+1)}$

2. Spacetime translation $\mathbb{R}^{1,3}$

(4-)momentum = generator of (spacetime) translation. $\mathbb{R}^{1,3} = \left\{ \begin{matrix} (p^0, p^1, p^2, p^3) \in \mathbb{R}^4 \\ p_\mu p^\mu = \eta_{\mu\nu} p^\mu p^\nu \end{matrix} \right\}$ is a Lie algebra w/

trivial bracket $[p, q] = 0 \ \forall p, q \in \mathbb{R}^{1,3}$.

If $P^\mu = \rho(p^\mu)$ is a irred. rep. of $\mathbb{R}^{1,3} \Rightarrow \hat{C}_2 = \eta_{\mu\nu} P^\mu P^\nu$ is the Casimir operator

($p^0 = (1,0,0,0)$, $p^1 = (0,1,0,0)$ …)

$\Rightarrow \hat{C}_2|\psi\rangle = m|\psi\rangle \ \forall |\psi\rangle \in \mathcal{H}$, i.e. $\underline{\text{the mass shell condition}}$! $\eta_{\mu\nu}P^\mu P^\nu = m\mathbb{1}$ $\begin{cases} m > 0 \\ m < 0 \\ m = 0 \end{cases}$

QFT by Weinberg: Particles are irred. rep. of $\underline{\text{Poincaré groups}}$ $\begin{cases} \text{Lorentz} \\ \text{translation} \end{cases}$

Wigner's little group method: (orbit method) 2^{nd} Casimir operator $W_\mu W^\mu = -\underset{\text{mass}}{m^2}\, \underset{\text{spin}}{s(s+1)}$ $\left(W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} \underbrace{J^{\nu\rho}}_{\text{generator of Lorentz group}} P^\sigma \right)$

$\Rightarrow$ Classifying particles by values of $m \Leftrightarrow$ mass shell or light cone, etc.

4. Others

$\begin{cases} \text{Fine structure of Hydrogen } (SO(4)) \\ \text{Landau level } (SU(1,1)) \\ \text{Running coupling constant in QCD } (SU(N)) \\ \vdots \end{cases}$

Many more applications!!