# A Brief Introduction to “Quaternion”

112022123 張秩宥

# What is Quaternion ?

- A number system extends the **complex number.**
- First described by the Irish mathematician Hamilton, so the algebra quaternion is often represented by ***H*** or $\mathbb{H}$ .
- Real number $\mathbb{R}$ : a
- Complex number $\mathbb{C}$: a+bi      a,b $\in \mathbb{R}$
- Quaternion $\mathbb{H}$: a+bi+cj+dk    a,b,c,d$\in \mathbb{R}$

# Definitions

- The $1$ $\mathbf{i}$ $\mathbf{j}$ $\mathbf{k}$ in quaternion are the fundamental quaternion units.
- Quaternion units have relation such that

  $$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}$$

  and we can know

  **ij** = - **ji = k** , **jk**= - **kj = i** , **ki** = - **ik = j** (they don't commute!)

# Definitions

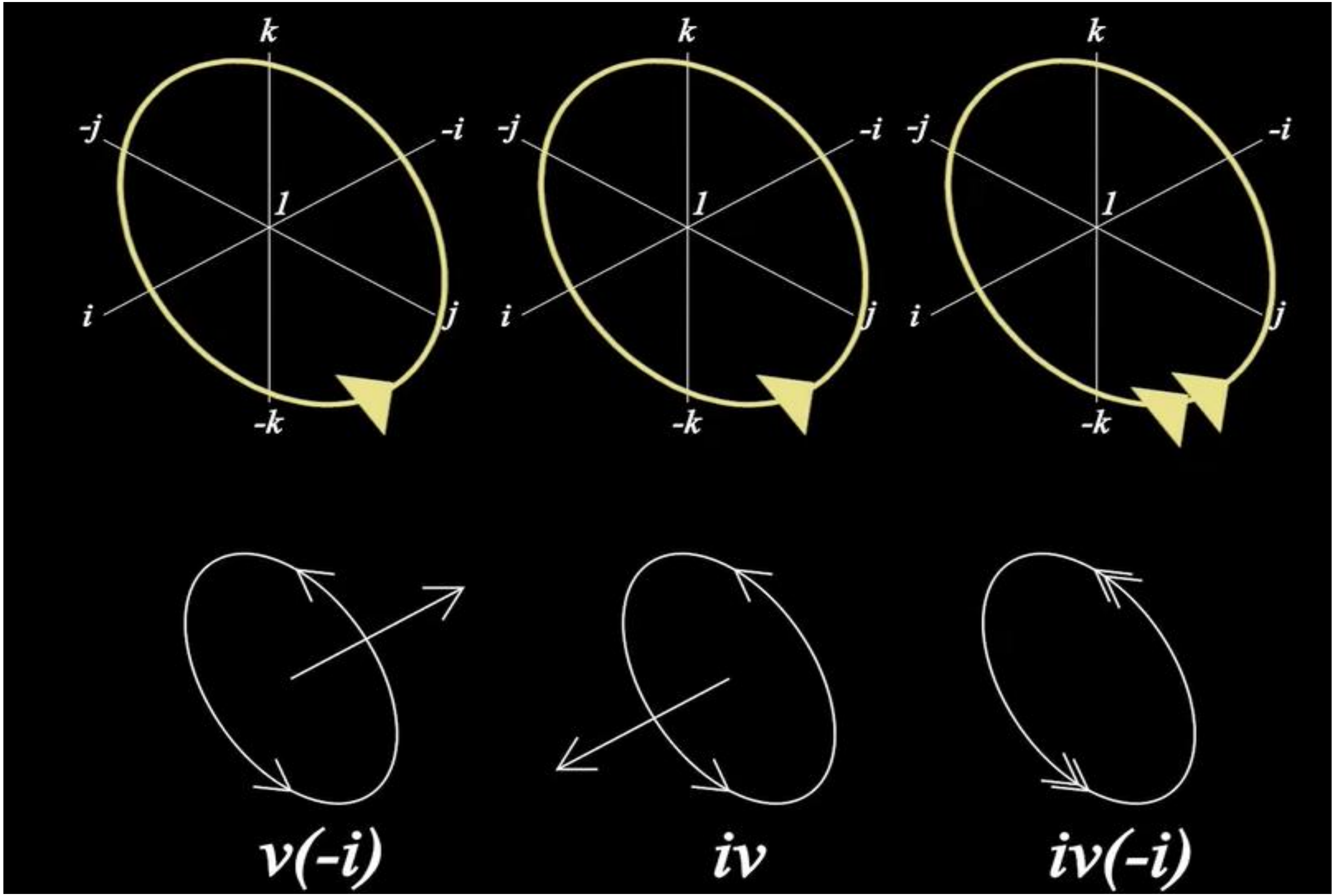
- q=a+bi+cj+dk≡[a,b,c,d]=[a,**v**], a:scalar part , **v**: vector part
- Conjugate: $q^* = [a, \boldsymbol{v}]^* \equiv [a, -\boldsymbol{v}]$, $(pq)^* = q^* p^*$
- Norm: $N(q) \equiv q * q^* = a^2 + \boldsymbol{v} \cdot \boldsymbol{v} = a^2 + b^2 + c^2 + d^2$
- Inverse: $q^{-1} = q^*/N(q)$
- Distributivity: $(p+q) * r = p * r + q * r$
- Compatibility with scalars: $(ap) * (bq) = (ab)(p * q)$

# Rotation

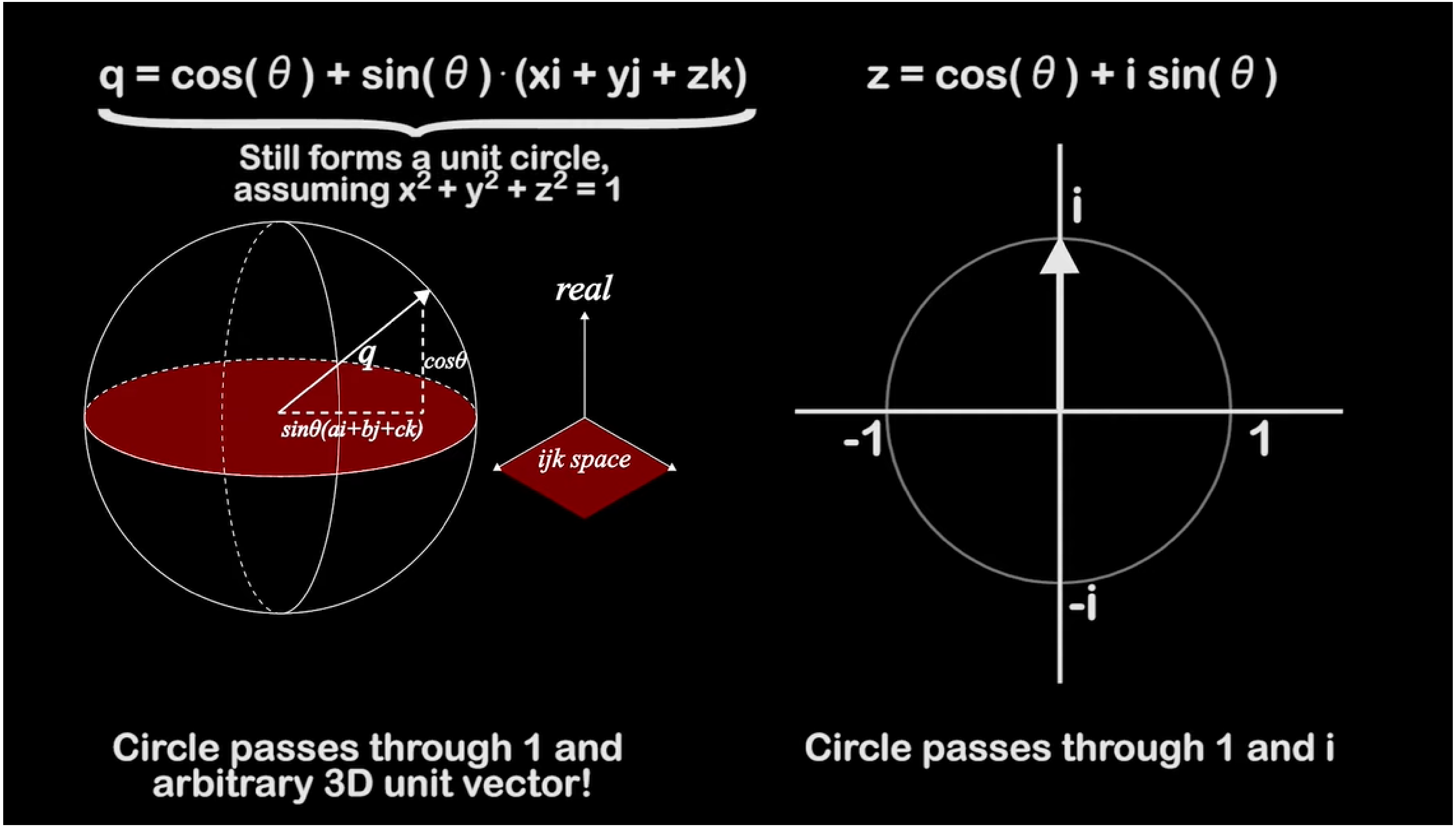
- A rotation of angle θ around **any** axis defined by unit vector $\hat{\boldsymbol{n}} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ can be described as

  $\boldsymbol{v}' = \boldsymbol{q}\boldsymbol{v}\boldsymbol{q}^*$, where $\mathrm{N}(\boldsymbol{q}) = 1$, $\boldsymbol{q} = \left[\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\hat{n}\right] = e^{\frac{\theta}{2}\hat{\boldsymbol{n}}}$
- Euclidean vectors (2,3,4)→ $2\hat{x} + 3\hat{y} + 4\hat{z}$

  →pure quaternion(0,2,3,4) (only vector part)
- We can use quaternion to describe the rotation in $\mathbb{R}^3$!




https://www.youtube.com/watch?v=jTgdKoQv738&t=214s

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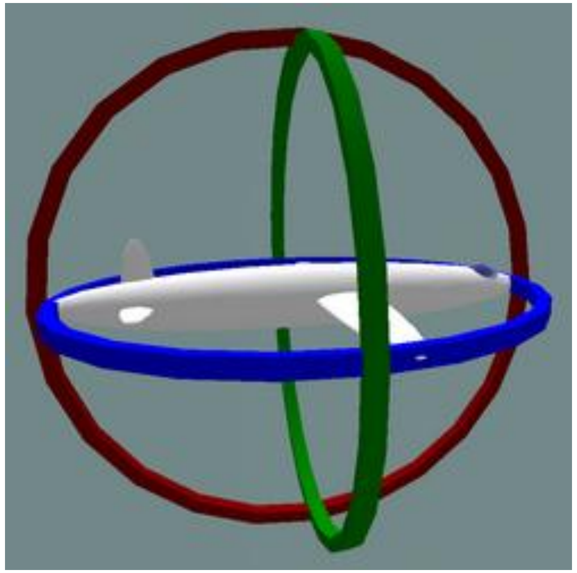
# Rotation

- Two rotations is given by $\boldsymbol{v}'' = \boldsymbol{q_2}(\boldsymbol{q_1 v q_1^*})\boldsymbol{q_2^*}$ (in quaternion way)
- The way we usually deal with the rotation in $\mathbb{R}^3$ of rigid body is using the rotation matrices and the Euler angles, the most general rotation matrix

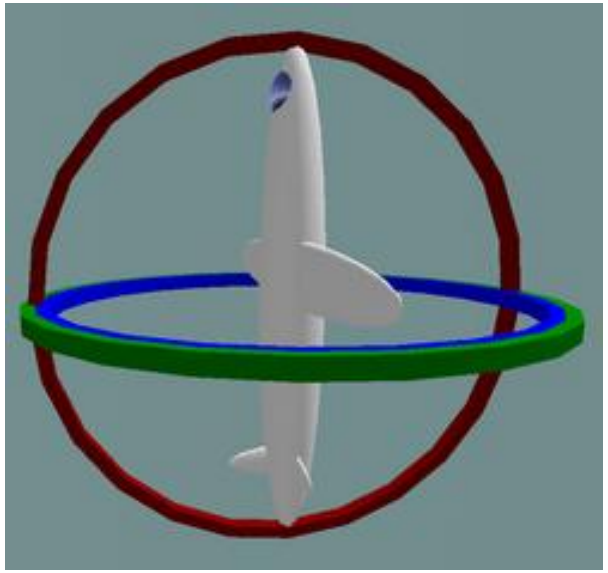
$$R = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Pros of Quaternion

- Quaternion gives us a faster, easier and more general way to write a rotation.
- The Euler angle may cause **"Gimbal lock".**



Normal situation



Gimbal lock

# Applications

- In Orbital Dynamics of Spacecraft
  - Using Quaternion to calculate the rotation is faster and can control the spacecraft better and smoother and avoid Gimbal lock.
  - Using in spacecraft attitude control.

    e.g. : Apollo project, satellites

- In QM
  - The SU(2) is isomorphic to group of unit quaternion group, so SU(2) can be mapped to the quaternion. Thus, we can find some unit quaternions correspond to the SU(2) metrices, and then describe the spin of particle.