Topological Space

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How to define a *topology*

Definition

A *topology* on a nonempty set X is collection of subsets of X, called *open sets*, such that:

(a) the empty set $\emptyset$ and the set X are open;

(b) the union of an arbitrary collection of open sets is open;

(c) the intersection of a finite number of open sets is open.

Example1(Indiscrete Topology)

When $X = \{1, 2, 3\}$, we can chose a *topology*, which the *open sets* of X

are X and $\emptyset$, which satisfy the definition.

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Example2(Discrete Topology)

With the same $X = \{1, 2, 3\}$, we can chose another *topology*, which the

open sets of X are $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}, \{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$,

which also satisfy the definition.

Definition

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Example3

When $X = \mathbb{R}$, the *open sets* of X consist of $\mathbb{R}$, $\emptyset$ and arbitrary unions of the following types:

- $(a, b)\,, \quad a, b \in \mathbb{R}$
- $(-\infty, a)\,, \quad a \in \mathbb{R}$
- $(a, \infty)\,, \quad a \in \mathbb{R}$

The collection of above *open sets* forms a *topology*.

The definition of *topological space*

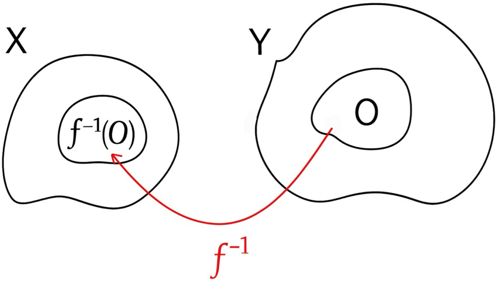
Definition

Let $\mathcal{T}$ be a topology on a set X. Then the pair $(X, \mathcal{T})$ is called a **topological space**. A subset $O \subseteq X$ is called an **open set** (or simply *open*) if $O \in \mathcal{T}$.

The continuous function

Definition

Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. A function $f : X \to Y$ is said to be **continuous** if for every open set $O \in \mathcal{S}$, the preimage $f^{-1}(O) \in \mathcal{T}$.



Definition of Homeomorphism

Definition

Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. They are said to be **homeomorphic** if there exists a function

$$f : X \to Y$$

such that:

(a) f is one-to-one and onto;

(b) both f and f^{-1} are **continuous**.

Such a function f is called a **homeomorphism**.

"This function maps one space to another without breaking its topological features, like openness and connectedness."

Different Physical Theories and Their Space Models

Theory	Space Model	Description
Newtonian Mechanics	$\mathbb{R}^3$	3D Euclidean space
Special Relativity (SR)	$\mathbb{R}^4$	4D flat spacetime
General Relativity (GR)	M	4D curved spacetime manifold

Definition of an n-dimensional Differentiable Manifold

Definition

A topological space $(M, \mathcal{T})$ is called an n-**dimensional differentiable manifold** if:

(a) There exists an open cover $\{O_\alpha\}$ of M such that for each α, there is a homeomorphism

$$\psi_\alpha : O_\alpha \to V_\alpha \subset \mathbb{R}^n$$

where V_α is an open subset of $\mathbb{R}^n$.

(b) If $O_\alpha \cap O_\beta \neq \emptyset$, then the transition map

$$\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(O_\alpha \cap O_\beta) \to \psi_\beta(O_\alpha \cap O_\beta)$$

is **smooth** (i.e., C^∞ differentiable).

"Each chart maps a local region of the manifold to an open subset of $\mathbb{R}^n$."

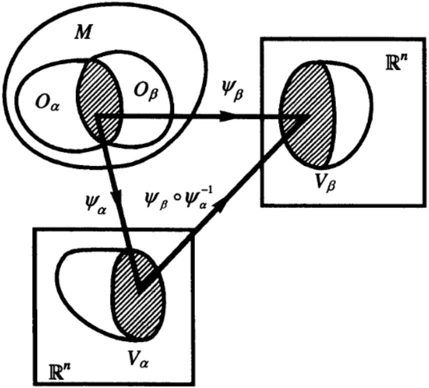


Figure: Visualizing a Differentiable Manifold

Why is Spacetime a Manifold in General Relativity?

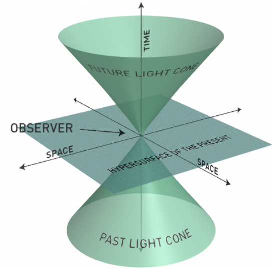
1. Spacetime in general relativity is **curved** by mass and energy.
2. A global coordinate system like $\mathbb{R}^4$ can't describe the whole spacetime.
3. We use a **4D smooth manifold** instead:
   - Locally: like flat $\mathbb{R}^4$
   - Globally: allows curvature and complex structure
4. This framework lets us define **tangent vectors**, **tensor fields**, and the **Einstein field equations**.

"Manifolds provide the stage for Einstein's theory of gravity."

Causal Structure and Light Cones

- Spacetime has both **topological** and **geometric** structures.
- The topology defines open sets to describe neighborhoods of events.
- The Lorentzian metric determines causal relationships between events.
- This gives rise to the **light cone**, which separates spacetime into:
  - **Future**: events that can be influenced.
  - **Past**: events that can influence the present.
  - **Elsewhere**: events with no causal connection.

Spacetime Diagram of a Light Cone



- The vertical axis is time; the horizontal axis is space.
- The light cone defines what can influence or be influenced.
- Outside the cone: events with no causal contact (faster-than-light required).