Moduli Space

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Outline

- What is Moduli Space ?
  - Definition
  - Example
- The relation to physics problems
- Summary

What is Moduli Space ?

What is Moduli Space ?

- Moduli space is about **Classification !**

- Moduli space can be thought of as **geometric** solutions to **geometric** classification problems.

- Moduli problem consists of three ingredients.
  - Objects, Equivalences and Families

Ingredients of a Moduli problem

- Objects
  - Which geometric objects would we like to **describe**, or **parametrize**?
- Equivalences
  - What do we mean about two object “the same” ?
- Families
  - How do we allow our objects to **change**, or **modulate**?

Example

- How can we classify triangles?

Example

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Now, let's go through the three ingredients of the moduli space one by one.

Example

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- Objects
  - Which geometric objects would we like to **describe**, or **parametrize**?

Example

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- Objects
  - Which geometric objects would we like to **describe**, or **parametrize**?

Of course, it's the triangle.

Example

- How can we classify triangles?
- Objects
  - **Triangle**

Example

- How can we classify triangles?
- Objects
  - **Triangle**
- Equivalences
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Example

- How can we classify triangles?
- Objects
  - **Triangle**
- Equivalences
  - What do we mean about two object “the same” ?

Did you notice that I didn't provide a classification "criterion" in the question?

Example

- How can we classify triangles?
- Objects
  - **Triangle**
- Equivalences
  - Here, we use **shape** as the criterion for classification.

Example

- How can we classify triangles?
- Objects
  - **Triangle**
- Equivalences
  - **Similar triangles**

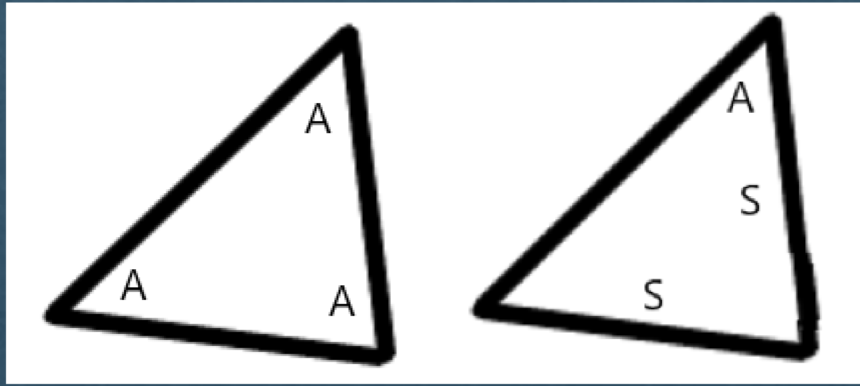
Example

- How can we classify triangles ?
- Objects
  - **Triangle**
- Equivalences
  - **Similar triangles**
- Families
  - How do we allow our objects to change, or modulate?

Example

- In fact, we learned this in junior high school.

- Similar triangle
  - AAA (Angle-Angle-Angle) triangle
  - SSA (Side-Side-Angle) triangle ($A < 90°$)
- The others are congruent (全等) triangles.



AAA SSA

Example

- Families
  - You can describe it using two angles or by knowing the ratio of two sides (q) and the included angle.

- The moduli space of all similar triangles
  - $\mathcal{M} = \{(\theta_1, \theta_2) | 0 < \theta_1 \leq \theta_2 < \pi, \theta_1 + \theta_2 < \pi\}$
  - $\mathcal{M} = \{(q, \theta) | \theta \in (0, \pi), q \in (0,1]\}$

Example

The definition of this is very simple.

- Families
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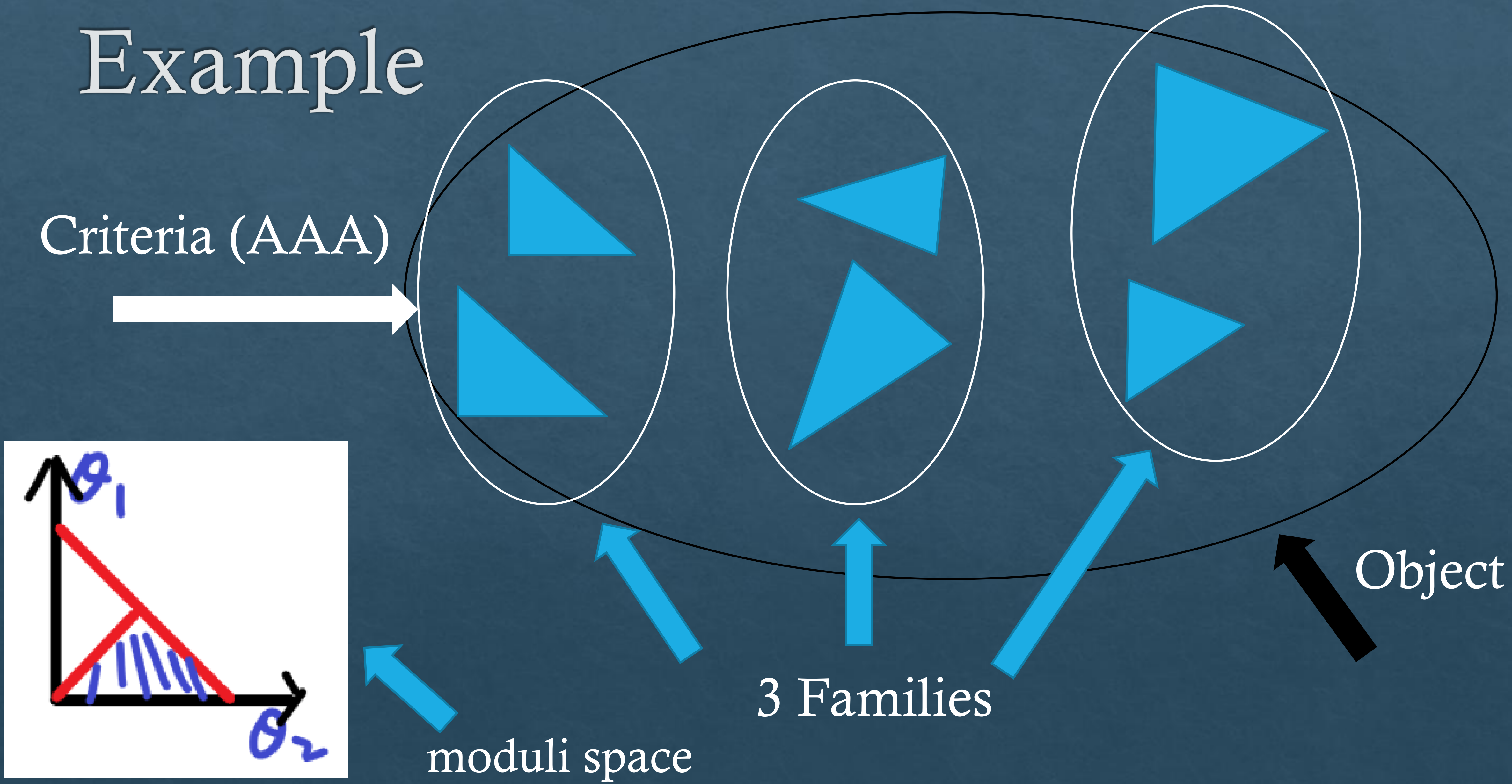
Example



The relation to physics problems

ϕ^4 kinks in 1 + 1 dimensions

In general, the Lagrangian density of such a scalar field theory is of the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi)$$

where $\phi = \phi(x, t)$ is a scalar field

After some calculation you can get

$$\partial_\mu\partial^\mu\phi + \frac{dU}{d\phi} = 0$$

Klein–Gordon type of field equation

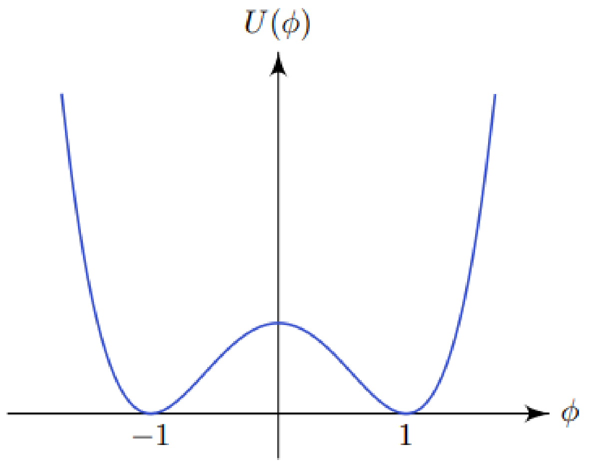
ϕ^4 kinks in 1 + 1 dimensions

But we are interested in a static solution. So the equation becomes

$$\frac{d^2\phi}{dx^2} = \frac{dU}{d\phi}$$

In ϕ^4 theory we can choose

$$U(\phi) = \frac{1}{2}(1 - \phi^2)^2$$



lectures by N. S. Manton and D. Stuart

Ginzburg–Landau theory

$$F(m) = \int \left[f_0 + \alpha m(x)^2 + \beta m(x)^4 + \left(\nabla m(x)\right)^2\right] dx$$

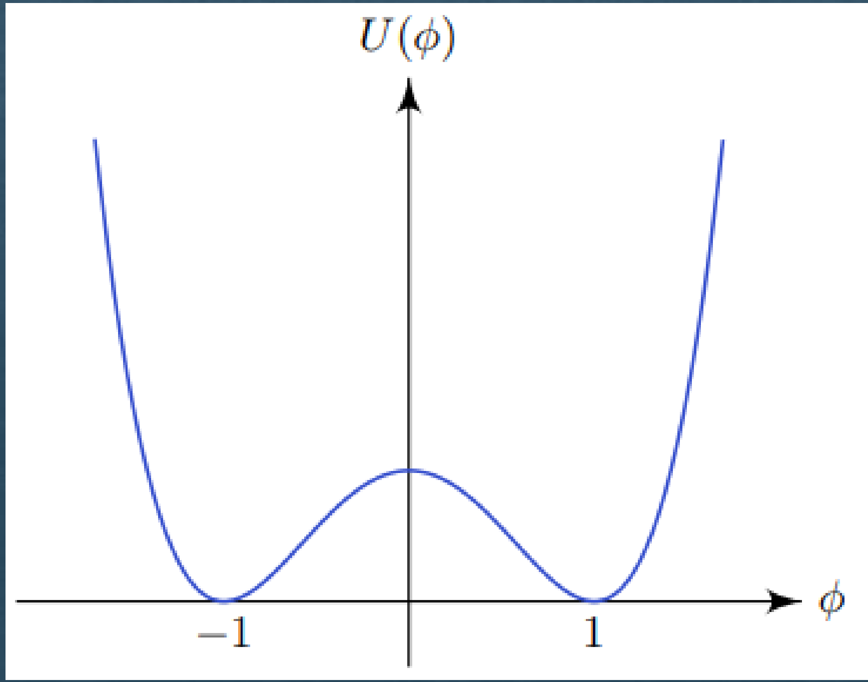
ϕ^4 kinks in $1 + 1$ dimensions

There are 2 vacua (lowest energy state)

Remember, ϕ is what you want to solve and it is a scalar field

The kink solution connects two different stable states (vacua) of the system. And it is ground state.

The field value gradually transitions from one vacuum state to another in space.



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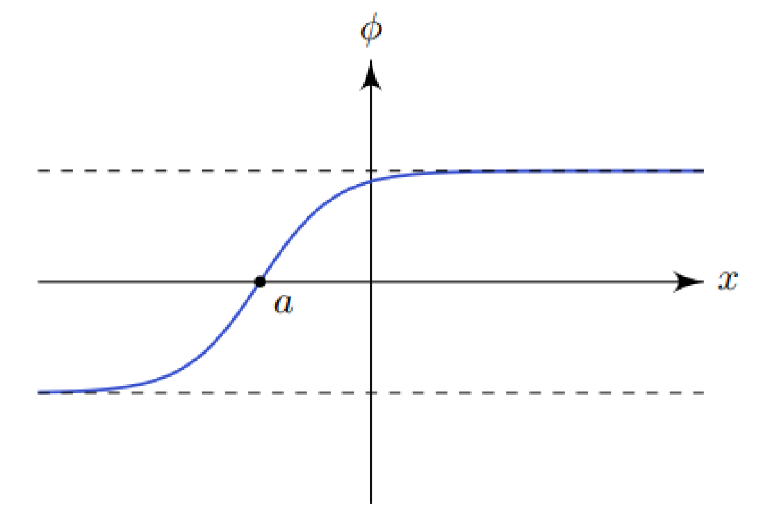
ϕ^4 kinks in 1 + 1 dimensions

After some calculation, you will get a solution like

$$\phi(x; a) = \tanh(x - a)$$

a is an arbitrary constant of integration



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Of course, there is also an anti-kink solution

$$\phi_{anti}(x; b) = -\tanh(x - b)$$

For all a and b correspond to the same energy. (Trust me)

ϕ^4 kinks in 1 + 1 dimensions

What is the moduli space in this case ?

ϕ^4 kinks in $1 + 1$ dimensions

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- Take the ϕ^4 kinks as our Object.

ϕ^4 kinks in $1 + 1$ dimensions

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- Take the energy as criteria.

ϕ^4 kinks in $1+1$ dimensions

What is the moduli space in this case ?

- Take the ϕ^4 kinks as our Object.

- Take the energy as criteria.

$$\mathcal{M} = \{a | a \in \mathbb{R}\}$$

- There are only one family (only one energy)

ϕ^4 kinks in $1 + 1$ dimensions

What are the advantages of using the Moduli space?

ϕ^4 kinks in $1 + 1$ dimensions

What are the advantages of using the Moduli space?

- When we want to look at non-static solutions
- You can think of it as the domain wall moving.

ϕ^4 kinks in $1 + 1$ dimensions

What are the advantages of using the Moduli space?

- When we want to look at non-static solutions
- You can think of it as the domain wall moving.

You can use this only when you think this is a Moduli Space

- We can write the moving kink as

$$\phi(x,t) = \tanh\big(x - a(t)\big)$$

Note that this required that da/dt is small.

ϕ^4 kinks in 1 + 1 dimensions

Recall that the Lagrangian also can be written as $L = T - V$

$$T = \int \frac{1}{2}\,\dot{\phi}^2\,dx, V = \int \left(\frac{1}{2}\phi'^2 + U(\phi)\right) dx$$

Where

$$\dot{\phi} = \frac{d\phi}{dt}, \phi' = \frac{d\phi}{dx}$$

Combine $\phi(x,t) = \tanh\big(x - a(t)\big)$, you will see

$$\dot{\phi} = \frac{d\phi}{dt} = -\phi'\,\frac{da}{dt}$$

ϕ^4 kinks in 1 + 1 dimensions

Trust me, the $V = \frac{4}{3}$ in this case, so here we only calculate T.

Combine $\dot{\phi} = -\phi' \frac{da}{dt}$ into T

$$T = \int \frac{1}{2} \dot{\phi}^2 \, dx = \frac{1}{2} \left(\frac{da}{dt} \right)^2 \int \phi'^2 \, dx = \frac{1}{2} M \left(\frac{da}{dt} \right)^2 = \frac{1}{2} M \dot{a}^2$$

Then the field Lagrangian becomes a particle Lagrangian

$$L = \frac{1}{2} M \dot{a}^2 - \frac{4}{3}$$

ϕ^4 kinks in $1+1$ dimensions

From

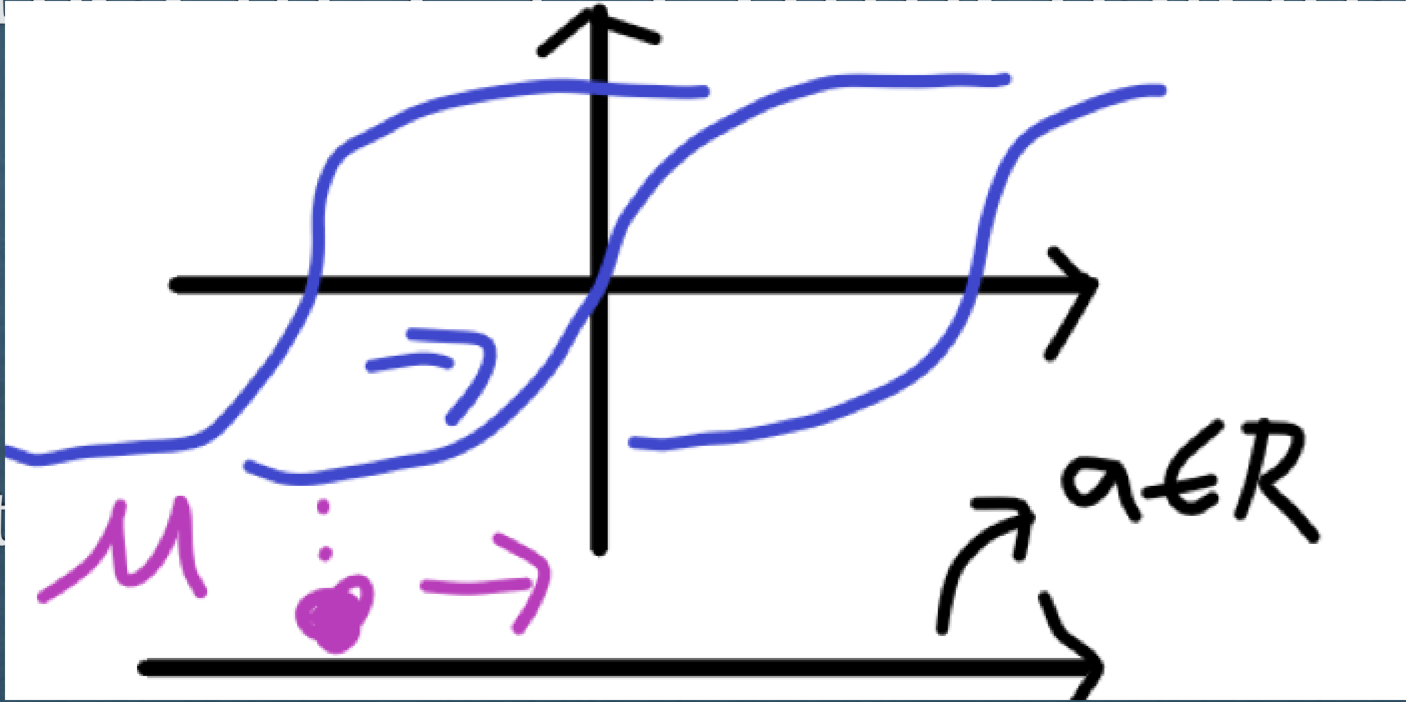
$$L = \frac{1}{2}M\dot{a}^2 - \frac{4}{3}$$

We can get

$$M\ddot{a} = 0 \rightarrow a = vt + Const.$$

ϕ^4 kinks in 1 + 1 dimensions

From



We can get

Did you notice that by introducing the moduli space here, the complex PDE is reduced to a simple ODE?

Phase Transition in Early Universe

- Beyond the standard model predicts the Higgs field (a scalar) to have different vacua, which could lead to a phase transition when part of the universe tunnels from the symmetric vacua to the Higgs vacua

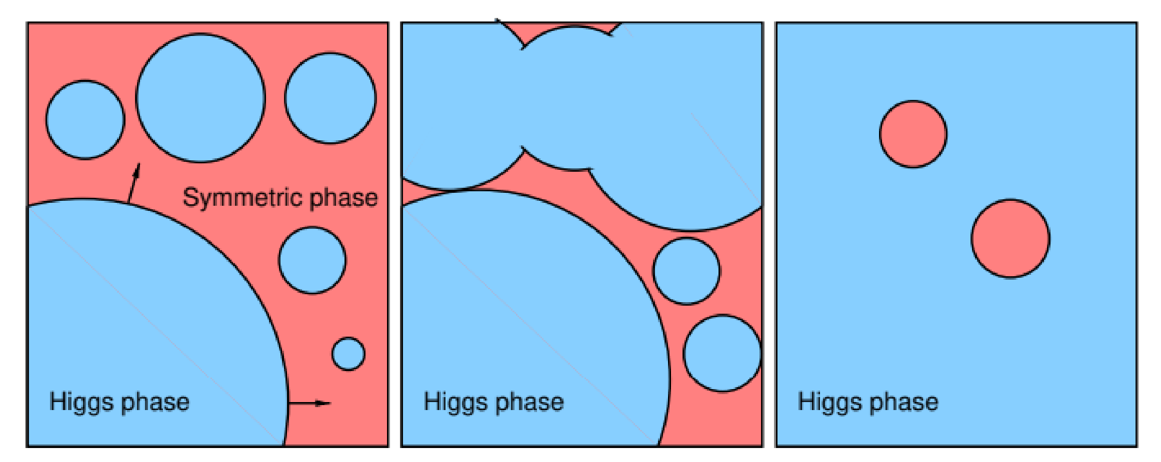
- The solution of the Higgs field between universe from the symmetric and Higgs phases (in spherical symmetry) can be modelled as a kink.

Phase Transition in Early Universe

- If the solution is spherically symmetric, then it would be the bubble and the boundary of bubble is the domain wall.

- To describe the motion of a domain wall, using the moduli space is an excellent choice.

Hindmarsh et al. 2021



Summary

- Moduli space is geometric solution
- Ingredients of Moduli Space
  - Objects
    - Which geometric objects would we like to **describe**, or **parametrize**?
  - Equivalences
    - What do we mean about two object “the same” ?
  - Families
    - How do we allow our objects to **change**, or **modulate**?

Summary

- Moduli Space in Physics
  - Transforming some complex partial differential field equation problems into ODEs on the moduli space allows us to understand certain dynamical properties more easily.

  - Magnetic domain wall or phase transition in early universe

Reference

- Classical and Quantum Soliton https://www.damtp.cam.ac.uk/user/examples/3P11e.pdf
- TASI Lectures on Solitons https://www.damtp.cam.ac.uk
- Moduli Spaces David D. Ben-Zvi
- Phase transitions in the early universe https://arxiv.org