RESIDUE THEOREM

物理系27級 蔡明傑

Outline

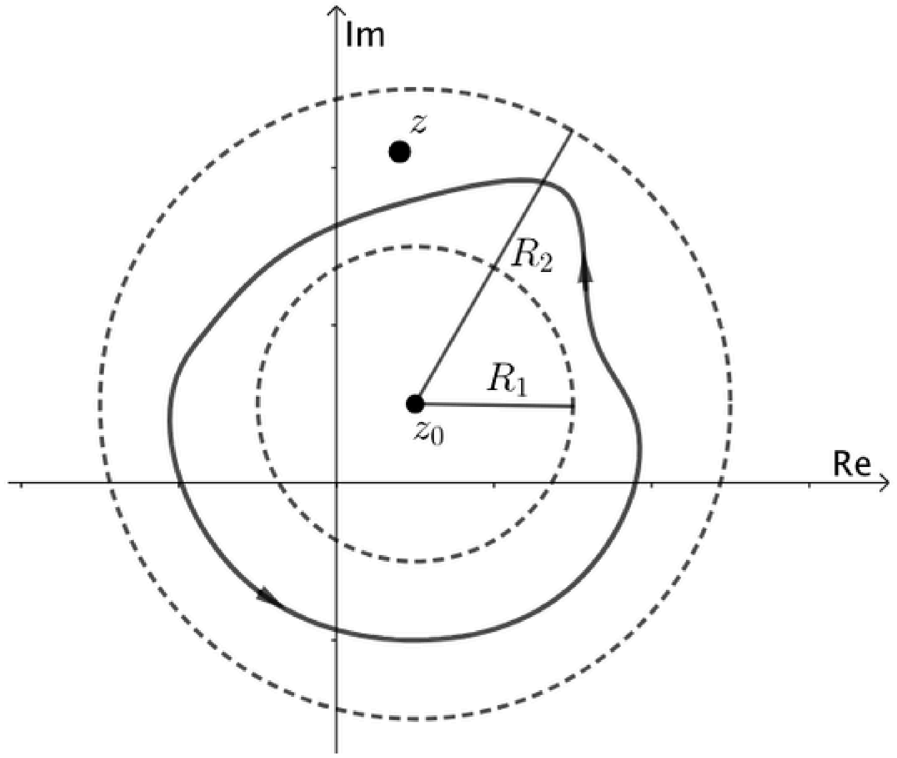
Residue theorem

ISOLATED SINGULARITIES & LAURENT SERIES

Definition: Isolated Singularity

For a function $f(z)$, the singularity z_0 is an **isolated singularity** if f is analytic on the deleted disk $0 < |z - z_0| < r$ for some $r > 0$.

ISOLATED SINGULARITIES & LAURENT SERIES



Theorem 5.3.1

Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centred at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \tag{5.3.1}$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^{n+1}}, \quad n = 0, 1, 2, \ldots \tag{5.3.2}$$

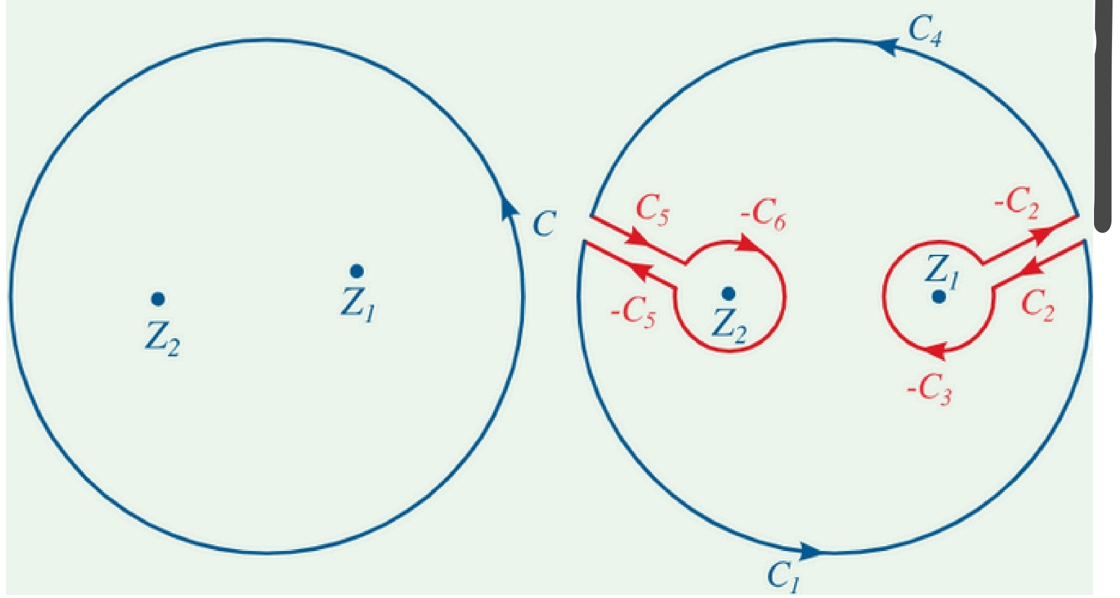
and

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^{-n+1}}, \quad n = 1, 2, \ldots \tag{5.3.3}$$

RESIDUE THEOREM $\oint_C f(z)\,dz = 2\pi i \sum_j R_j$

$$\int_{\tilde{C}} f(z)\,dz = \int_{C_1+C_2-C_3-C_2+C_4+C_5-C_6-C_5} f(z)\,dz = 0$$

$$\tilde{C} = C_1 + C_2 - C_3 - C_2 + C_4 + C_5 - C_6 - C_5$$



$$\int_{C_1+C_4} f(z)\,dz = \int_{C_3+C_6} f(z)\,dz$$

$$f(z) = \ \ldots + \frac{b_2}{(z-z_1)^2} + \frac{b_1}{z-z_1} + a_0 + a_1(z-z_1) + \ \ldots$$

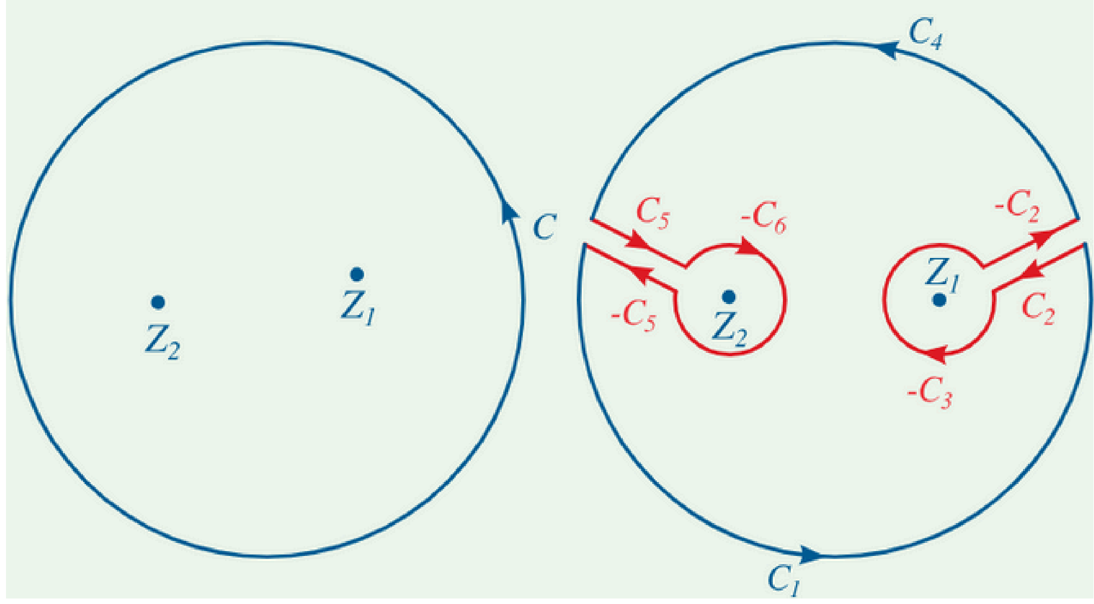
RESIDUE THEOREM $\oint_C f(z)\,dz = 2\pi i \sum_j R_j$

$$\int_{C_3} f(z)\,dz = \int_{C_3} \ldots + \frac{b_2}{(z-z_1)^2} + \frac{b_1}{z-z_1} + a_0 + a_1(z-z_1) + \ldots dz$$
$$= 2\pi i b_1$$
$$= 2\pi i \text{Res}(f, z_1)$$

$$\int_{C_6} f(z)\,dz = 2\pi i \text{Res}(f, z_2)$$

$$\int_C f(z)\,dz = 2\pi i[\text{Res}(f, z_1) + \text{Res}(f, z_2)]$$

$$\tilde{C} = C_1 + C_2 - C_3 - C_2 + C_4 + C_5 - C_6 - C_5$$



Theorem 9.5.1 Cauchy's Residue Theorem

Suppose $f(z)$ is analytic in the region A except for a set of isolated singularities. Also suppose C is a simple closed curve in A that doesn't go through any of the singularities of f and is oriented counterclockwise. Then

$$\int_C f(z)\,dz = 2\pi i \sum \text{residues of } f \text{ inside } C$$

APPLICATION 1: GREEN'S FUNCTION

Define the Green function

$$\left(\frac{d^2}{dx^2} - a^2\right) G(x) = \delta(x)$$

By using fourier transform

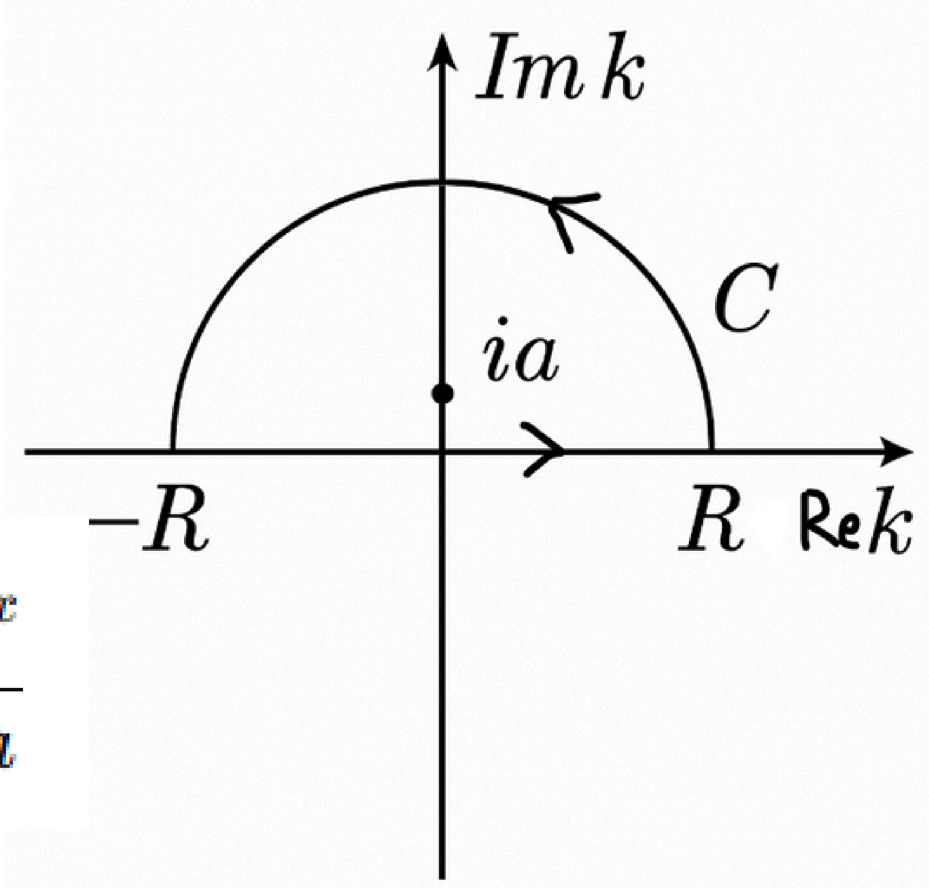
$$(-k^2 - a^2)\tilde{G}(k) = 1$$

By using inverse-fourier transform

$$G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(k) e^{ikx}\, dk = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2 + a^2}\, dk$$

APPLICATION 1: GREEN'S FUNCTION

$$G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(k)e^{ikx}\,dk = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2+a^2}\,dk$$



Here, we can calculate the integral by Complex analysis and Residue Theorem

$$\oint_C f(k)\,dk = 2\pi i \cdot \mathrm{Res}_{k=ia}$$

And we can calculate the residue

$$\mathrm{Res}_{k=ia}\left(\frac{e^{ikx}}{k^2+a^2}\right) = \lim_{k\to ia}(k-ia)\cdot\frac{e^{ikx}}{(k-ia)(k+ia)} = \frac{e^{iax}}{2ia}$$

We can write down the solution

$$G(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ikx}}{k^2+a^2}\,dk \Rightarrow G(x) = -\frac{1}{2\pi}\cdot\frac{\pi}{a}e^{-a|x|} = -\frac{1}{2a}e^{-a|x|}$$

APPLICATION 2: SCATTERING AMPLITUDE

$$\mathcal{A}(s) = \frac{g^2}{s - M^2 + i\epsilon}$$

In this formula,
s is the squared center-of-mass energy of the incoming particles.
M is the mass of an intermediate particle.
g is Coupling constant (interaction strength)
$i\epsilon$ Infinitesimal imaginary part ensuring causality and correct pole contour

APPLICATION 2: SCATTERING AMPLITUDE

$$\mathrm{Res}_{s=M^2}\mathcal{A}(s) = g^2$$

THANK YOU