The Power of Fourier Transform

PHYS26 CHUN-TUNG,TSAI(蔡竣同)

Outline

- Fourier Series
- Fourier Transform(FT)
- Application in Physics
- Discrete Fourier Transform(DFT)

Fourier series

- Any periodic function can be expanded in terms of sine and cosine function:

- $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega t) + B_n \sin(n\omega t)\,, \text{for } t_0 < t < t_0 + T$

- ω is fundamental angular frequency.
- The coefficient a_0 、A_n 、B_n can be calculated by the inner product , because the orthogonal property with each basis.
- $t \Rightarrow x, \omega \Rightarrow k$

Fourier Transform

- For non-periodic function, how can we know the frequency information? we take the period to infinity!

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

- Represents a function as a continuous spectrum of frequencies
- Applicable in signal analysis, physics, and engineering

Solving the Heat Equation Using FT

Heat Equation:

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 (x,t)}{\partial x^2}$$

with initial Condition $u(x,0) = f(x)$

Apply FT over x :

$$\hat{u}(k,t) = \mathcal{F}\{u(x,t)\} \Rightarrow \frac{\partial \hat{u}(k,t)}{\partial t} = -\alpha k^2 \hat{u}(k,t)$$

Solve for ODE: $\hat{u}(k,t) = \hat{f}(k) \cdot e^{-\alpha k^2 t}$, which $\hat{f}(k) = \mathcal{F}\{f(x)\}$

Inverse FT: $u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k) \cdot e^{-\alpha k^2 t} e^{ikx} dk$

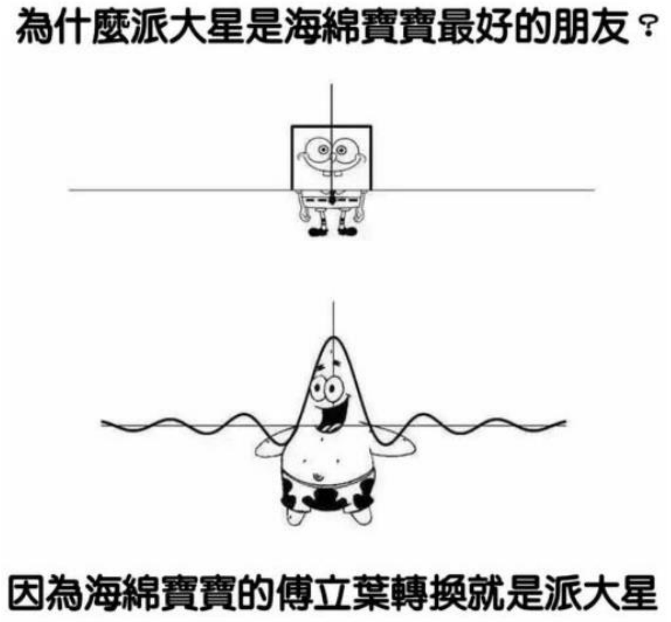
High frequency decay faster because to $e^{-\alpha k^2 t}$term

FT in Quantum Physics(1/2)

- Position $\psi(x)$and momentum$\phi(p)$ wavefunctions are Fourier pairs

$$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi\,(k) \cdot e^{-ipx/\hbar} dx$$

- Narrow in space(or t) ⇒wide in momentum(or f)
- Leads to uncertainty relation: $\Delta x \Delta p \geq \frac{\hbar}{2}$



FT in Quantum Physics(2/2)

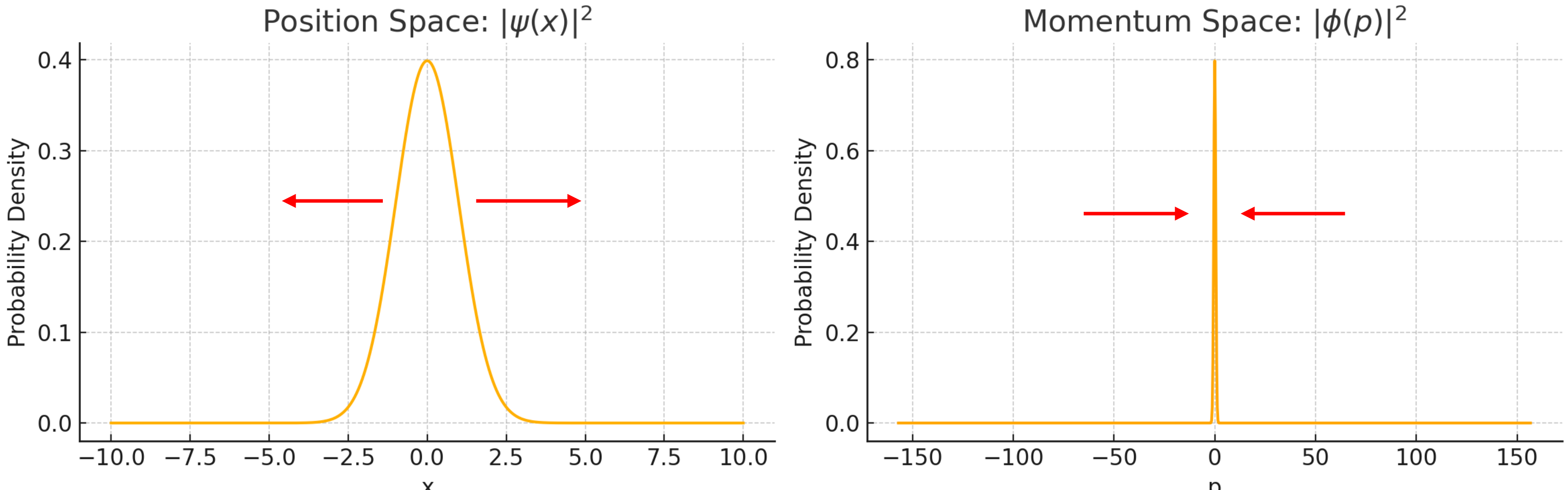
Let the wavefunction be a normalized Gaussian:

$$\psi(x) = \frac{1}{\sqrt[4]{2\pi\sigma_x^2}} e^{-x^2/(4\sigma_x^2)}$$

Then its Fourier transform is:

$$\phi(p) = \frac{1}{\sqrt[4]{2\pi\sigma_p^2}} e^{-p^2/(4\sigma_p^2)}$$

Which $\sigma_p = \frac{\hbar}{2\sigma_x}$



FT in FM and Sound Mixing

- Sound signals can be decomposed into frequencies
- FT reveals the components of complex tones
- FM: Frequency Modulation spreads frequency spectrum
- Mixing: overlapping signals in time ↔ additive spectra in frequency

Limitation of FT

- FT assumes linear time-invariant (LTI) systems
- Fails in nonlinear systems: new frequencies generated
- No simple "multiply in frequency domain" behavior
- Still useful in approximate/linearized analysis

Discrete Fourier Transform

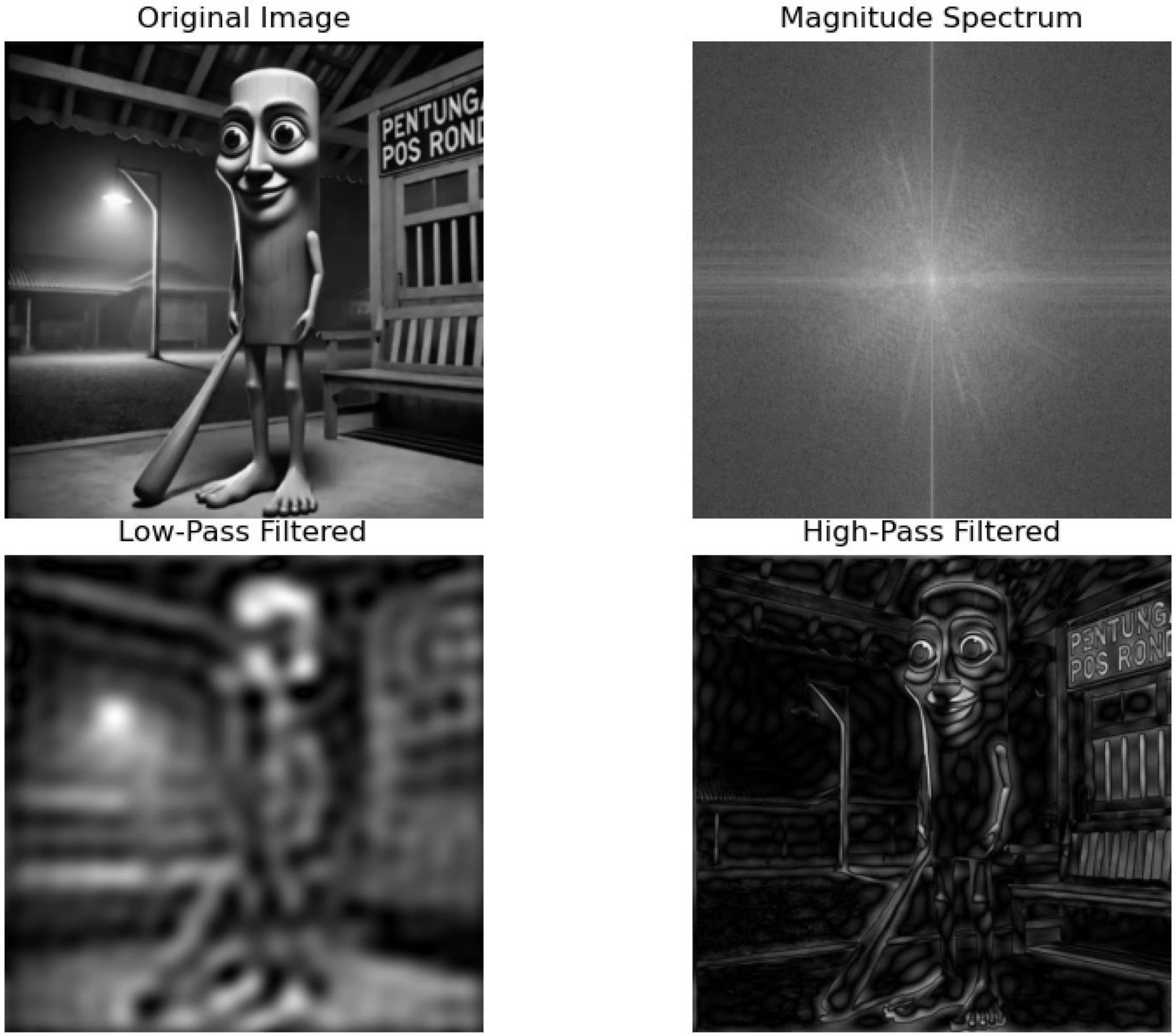
- The Discrete Fourier Transform is used for signals represented as arrays of discrete data points (e.g., images).
- 2D DFT Equation:
  - Forward Transform:

$$F\left(k_p, k_q\right) = \Delta x \Delta y \sum_{m,n} f(x_m, y_n) e^{-i(k_p x_m + k_q y_n)}$$

  - Inverse Transform:

$$F(x_m, y_n) = \frac{\Delta k_x \Delta k_y}{(2\pi)^2} \sum_{p,q} f(k_p, k_q) e^{i(k_p x_m + k_q y_n)}$$

Example


Original Image
Magnitude Spectrum
Low-Pass Filtered
High-Pass Filtered

Thanks for listening