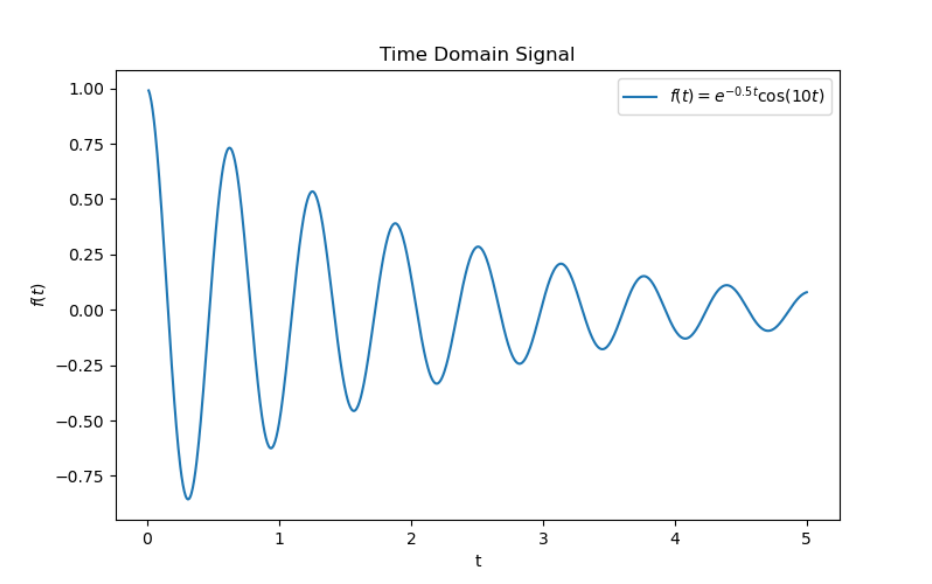
Laplace Transform

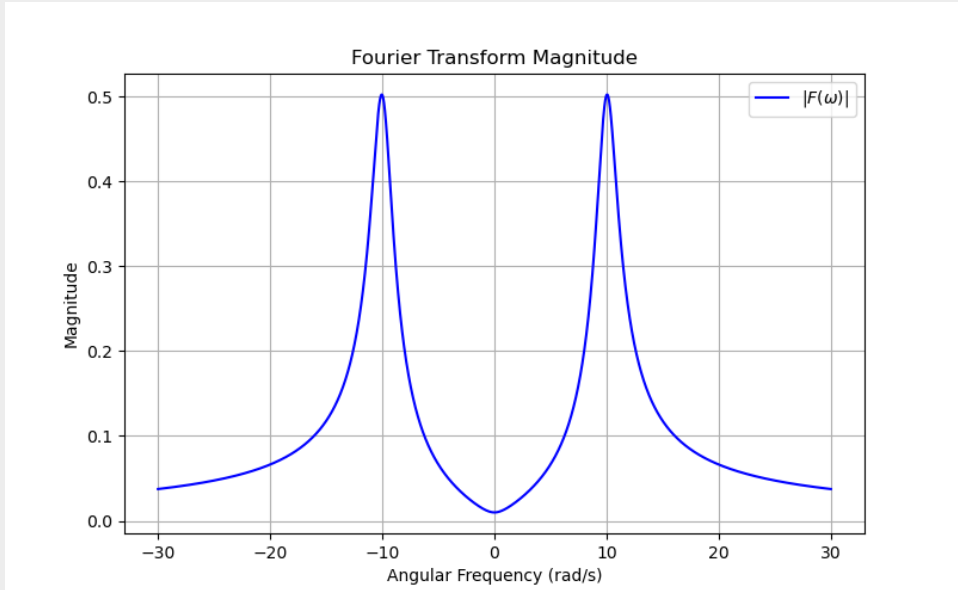
Fourier Transform

- Definition: $f(t) \quad \rightarrow \quad F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

- Examples:

$$f(t) = e^{0.5t}\cos(10t) \quad \rightarrow \quad \text{peak at } \omega = 10$$





But what about the information of decay rate ?

Laplace Transform

- Definition: $f(t) \rightarrow F(s) = \int_0^\infty f(t)e^{-st}dt,$ where $s = \sigma + i\omega$

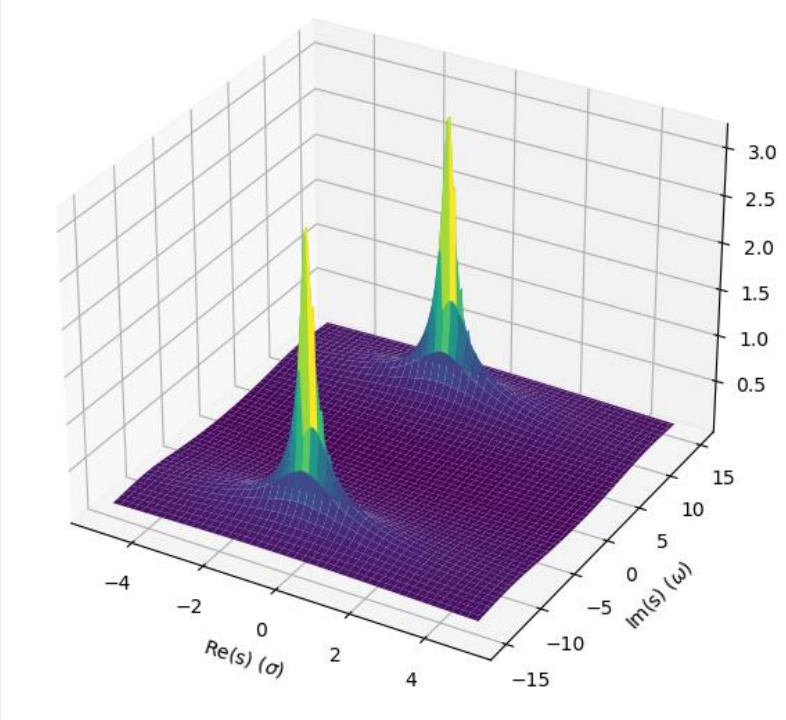
$$= \int_0^\infty f(t)e^{-(\sigma+i\omega)t}dt$$

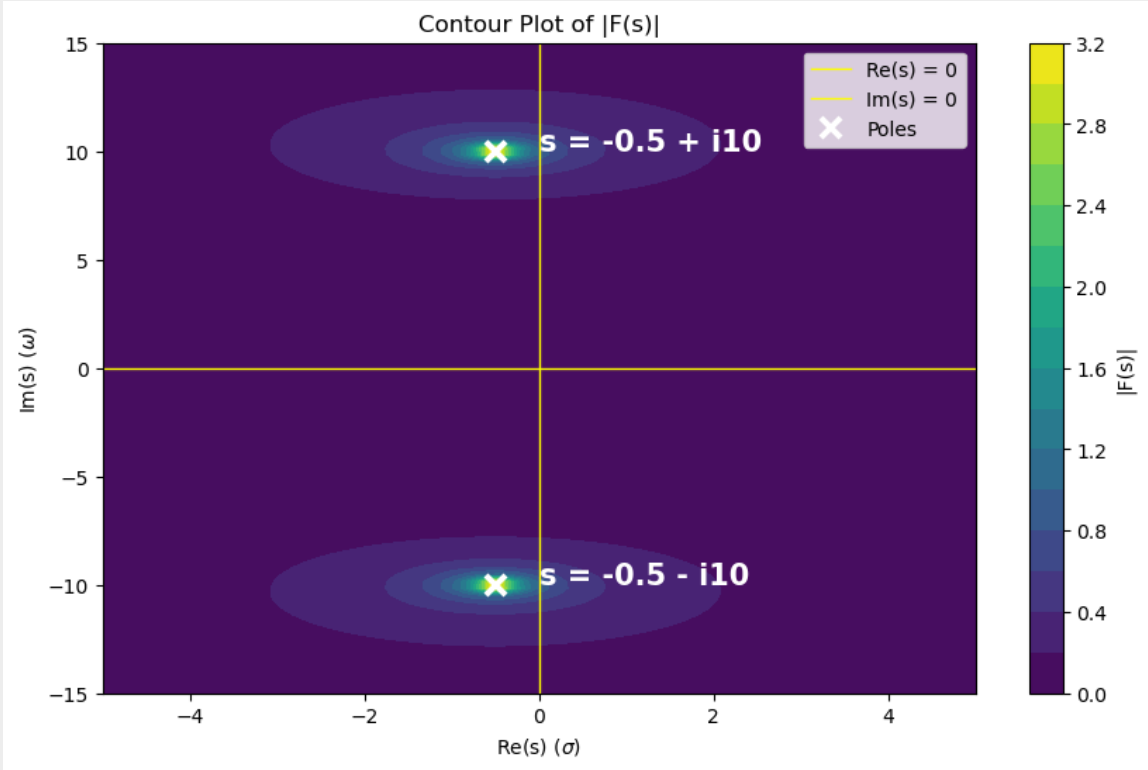
$$= \int_0^\infty [f(t)e^{-\sigma t}]e^{-i\omega t}dt$$

- Laplace transform is the Fourier transform of $[f(t)e^{-\sigma t}]$ 

Laplace Transform : Examples

$$f(t) = e^{0.5t}\cos(10t) \quad \rightarrow \quad F(s) = \frac{s + 0.5}{(s + 0.5)^2 + 10^2}$$





- Pole

  real part: exponential growth imaginary part: frequency

  <0 stable , >0 unstable

Laplace Transform: Properties

- Linearity:

$$\alpha f(t) + \beta g(t) \rightarrow \alpha F(s) + \beta G(s)$$

- Differentiation Property:

$$f(t) \rightarrow F(s)$$

$$f'(t) \rightarrow sF(s) + f(0)$$

$$f''(t) \rightarrow s^2F(s) + sf(0) + f'(0)$$

✓ These properties help us solve Differential Equation

Application in Physics : Solve Differential Equation

Example: Damped Forced Oscillator

$$m\ddot{x} + \gamma\dot{x} + kx = \mathrm{F(t)}$$

$$\ddot{x} + 3\dot{x} + 2x = u(t-1) - u(t-2)\,, x(0) = 0, \dot{x}(0) = 0$$

where $u(t)$ is a unit step function

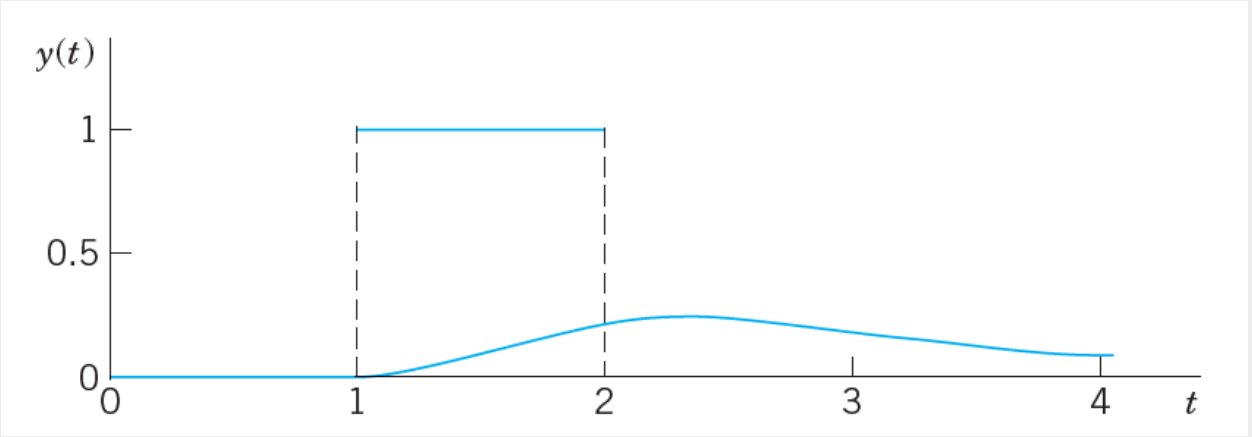
$Laplace\ transform\ both\ sides$:

$$s^2X + 3sX + 2X = \frac{e^{-s} - e^{-2s}}{s}$$

$$X(s) = \frac{1}{s(s^2 + 3s + 2)}(e^{-s} - e^{-2s})$$

$Inverse\ Laplace\ transform$:

$$x(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$



Square wave input and response

Advantages of the Laplace Transform Method

1. Simplify PDE -> ODE ; ODE -> Algebraic equation
2. Naturally incorporates initial and boundary conditions
3. Handles discontinuous inputs and impulsive inputs
4. Transforms time-domain problems into s-domain

Other Applications:

- Electrical Circuits:

  ~ simplify complicated circuits into algebraic equation

  ~ handles circuits with switch or time-varying sources

- Control Systems

  ~ compute transfer function to describe how systems respond to inputs

  ~ Analyze stability and design feedback loops in s-domain