Green’s Function

Solving differential equations

How to solve DEs with green's function？

- $\hat{\mathcal{L}}\, y(x) = f(x)$
- $\hat{\mathcal{L}}\, G(x,s) = \delta(x-s)$
- $y(x) = \int_{-\infty}^{\infty} f(s) G(x,s)\, ds$

Why does $y(x) = \int_{-\infty}^{\infty} f(s) G(x,s)\, ds$?

- $\hat{\mathcal{L}} \int_{-\infty}^{\infty} f(s) G(x,s)\, ds$

$= \int_{-\infty}^{\infty} \hat{\mathcal{L}} f(s) G(x,s)\, ds$

$= \int_{-\infty}^{\infty} f(s)\, \hat{\mathcal{L}} G(x,s)\, ds$

$= \int_{-\infty}^{\infty} f(s) \delta(x-s)\, ds = f(x)$

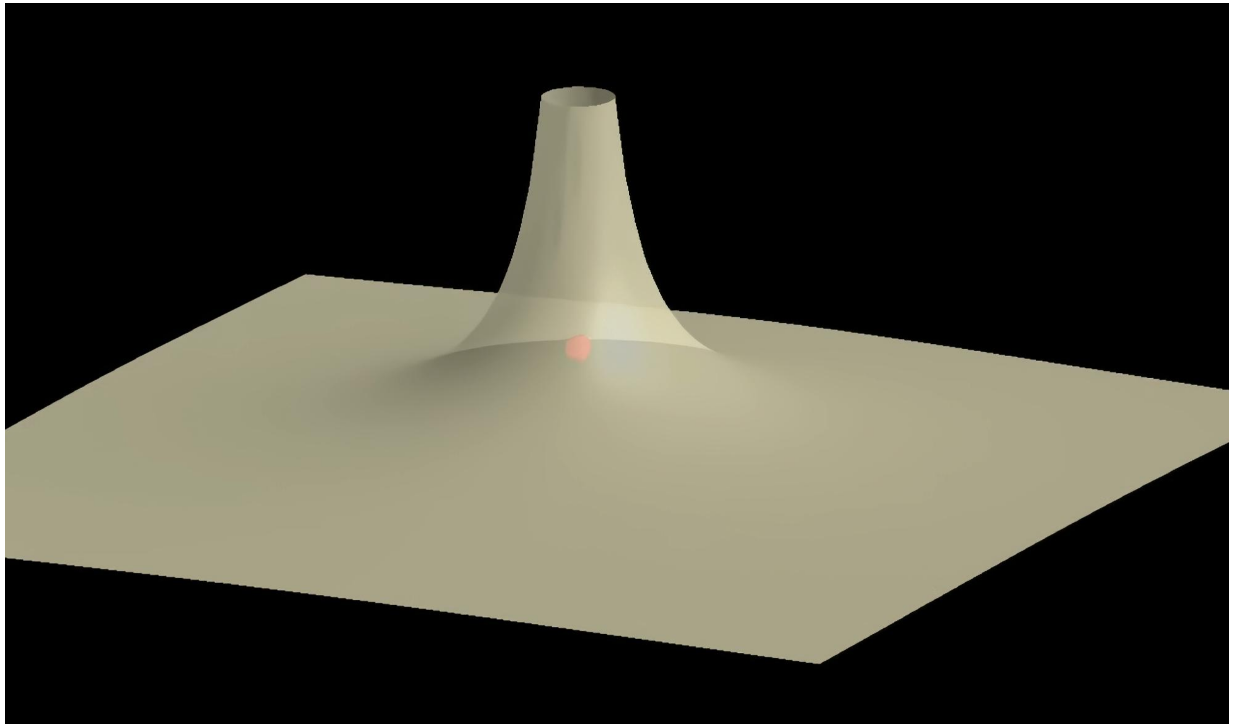
Poisson Equition

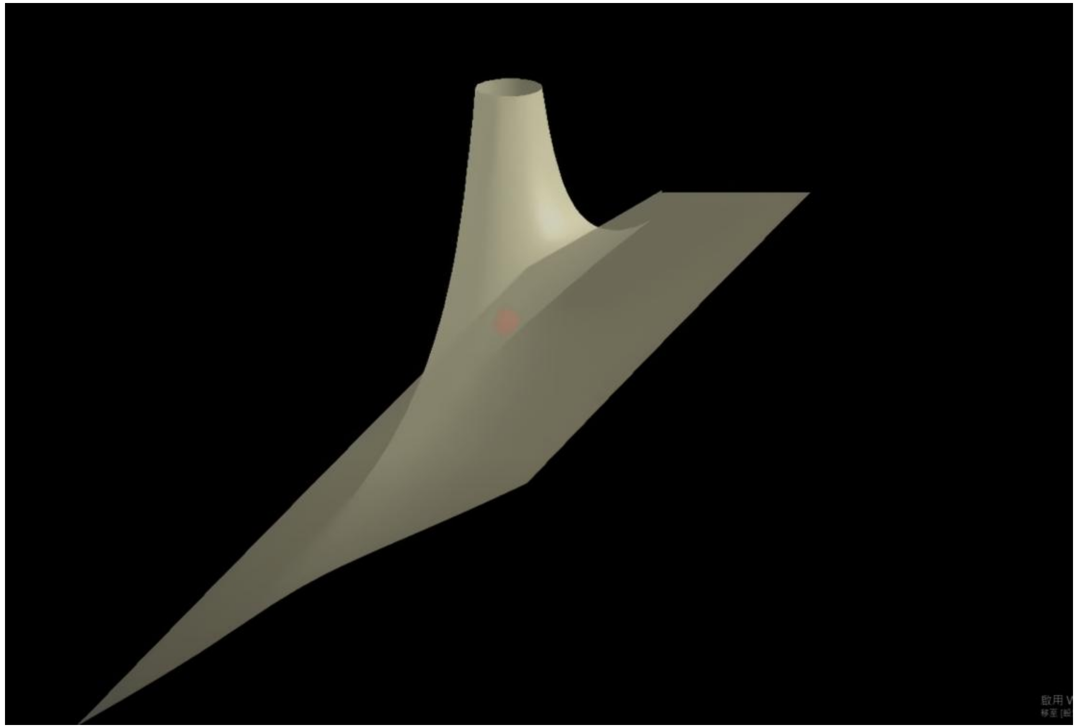
- $\nabla^2 V(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0}$
- BC: $V(\vec{r}) = h(\vec{r}),\ on\ S$

- $\nabla^2 G(\vec{r}, \vec{s}) = \delta(\vec{r} - \vec{s})$

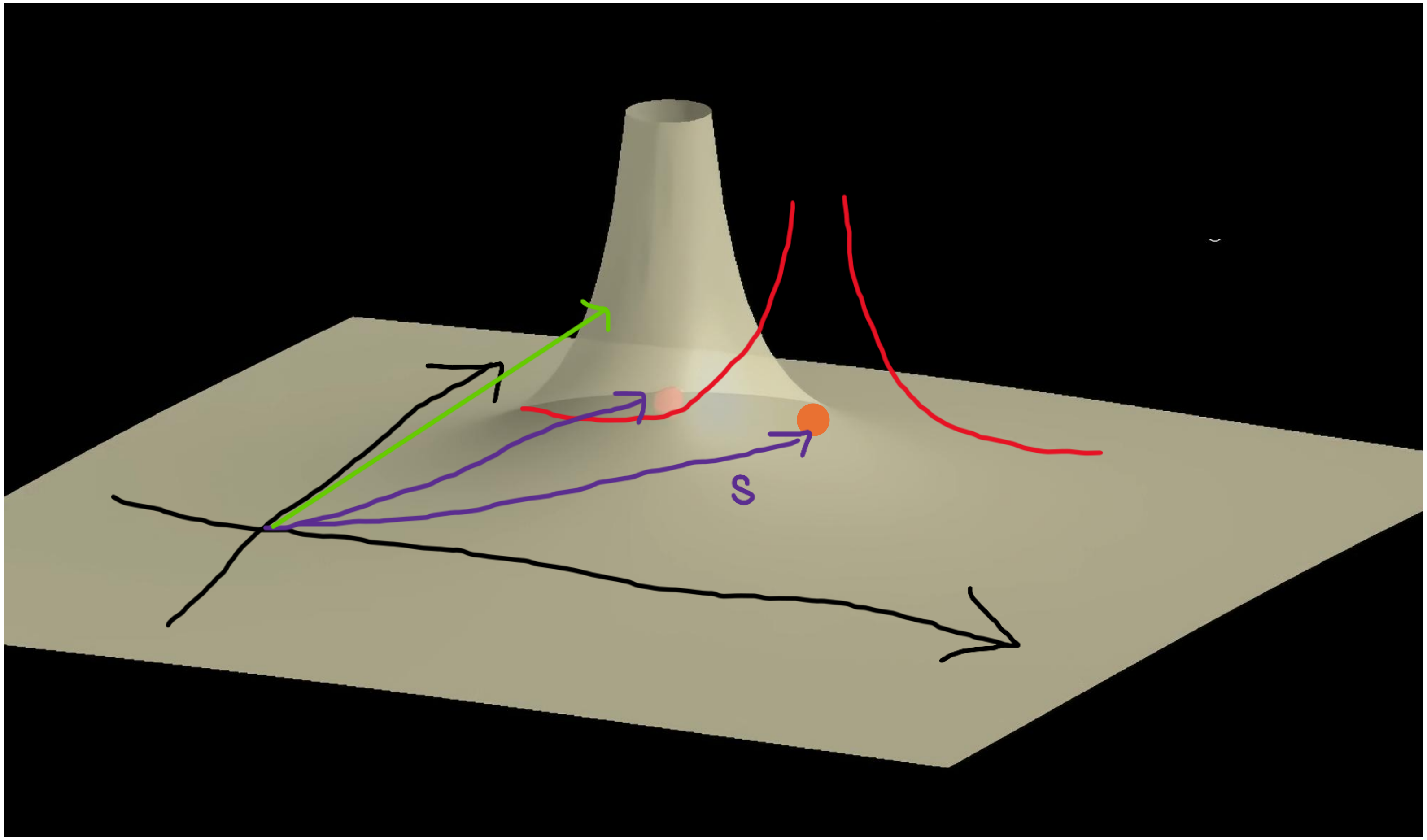
- $G(\vec{r}, \vec{s}) = \frac{1}{4\pi\ (\vec{r} - \vec{s})}$ under homogeneous BC
- $V(\vec{r}) = \int_{-\infty}^{\infty} \frac{\rho(\vec{s})}{\varepsilon_0} G(\vec{r}, \vec{s}) ds$

Boundary condition





Picture of green's function in electrical potential



Green's identities

- Divergence theorem : $\oint_{\partial\mathbb{U}} F \cdot n \, dA = \int_{\mathbb{U}} \nabla \cdot F \, dV$
- Let $F = v\nabla u$
- $\oint_{\partial\mathbb{U}} v\nabla u \cdot n \, dA = \int_{\mathbb{U}} \nabla \cdot (v\nabla u) dV$

$= \oint_{\partial\mathbb{U}} v \cdot \frac{\partial u}{\partial n} dA = \int_{\mathbb{U}} \nabla v \nabla u + v\nabla^2 u \, dV \quad -(G1)$

- $\oint_{\partial\mathbb{U}} v \cdot \frac{\partial u}{\partial n} - u \cdot \frac{\partial v}{\partial n} dA = \int_{\mathbb{U}} v\nabla^2 u - u\nabla^2 v dV \quad -(G2)$

Green's identities

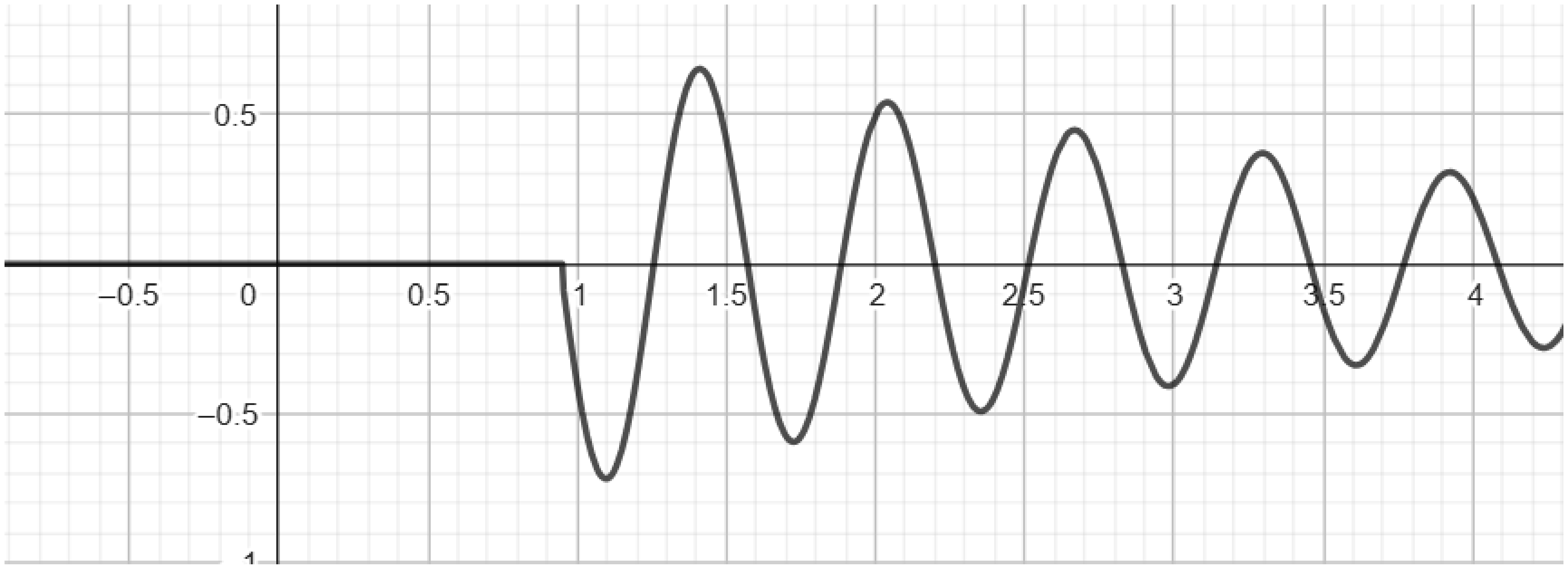
- $\oint_{\partial\mathbb{U}} v \cdot \frac{\partial u}{\partial n} - u \cdot \frac{\partial v}{\partial n} dA = \int_{\mathbb{U}} \ v\nabla^2 u - u\nabla^2 v dV \ \ - (G2)$
- $v = G(\vec{r}, \vec{s}), \ \mathrm{u} = V(\vec{r})$

- $\oint_{\partial\mathbb{U}} G \frac{\partial V}{\partial n} - h \frac{\partial G}{\partial n} dA = \int_{\mathbb{U}} \ G \frac{\rho}{\varepsilon_0} - V\delta(\vec{r} - \vec{s}) \, dV = \int_{\mathbb{U}} \ G \frac{\rho}{\varepsilon_0} \, dV - V(\vec{r})$

$\rightarrow V(\vec{r}) = \int_{\mathbb{U}} \ G \frac{\rho}{\varepsilon_0} \, dV + \oint_{\partial\mathbb{U}} h \frac{\partial G}{\partial n} - G \frac{\partial V}{\partial n} dA$

Underdamped harmonic oscillator

- $\left(\partial_t^2 + 2\gamma + \partial_t \omega_0^2\right) x(t) = F(t)$
- $G(t,s) = \Theta(t-s) e^{-\gamma(t-s)} \frac{\sin[\omega(t-s)]}{\omega}, \ \omega = \sqrt{\omega_0^2 - \gamma^2}$



One dimensional wave equation

- $(\partial_t^2 - v^2 \partial_x^2)\mathrm{y}(\mathrm{x}, \mathrm{t}) = 0$

- $G(\vec{r}, \vec{r}', t, t') = \frac{1}{2c} \delta[(t - t') - \frac{\vec{r} - \vec{r}'}{v}]$