Berry Phase

May 14, 2025

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Consider Hamiltonian

$$H\left(\lambda^i(t)\right)$$

where $\lambda^i(t)$ are time-dependent parameters.

Prepare the system in ground state and vary $\lambda(t)$ slowly along a closed loop in the parameter space. By adiabatic theorem,

$$|\psi(\lambda(t))\rangle \rightarrow e^{i\gamma}\left|\psi\left(\lambda(t)\right)\right\rangle$$

where $e^{i\gamma}$ is the phase difference, contributed by dynamical phase $e^{i\frac{E}{\hbar}t}$ and Berry phase $e^{i\gamma_n(t)}$

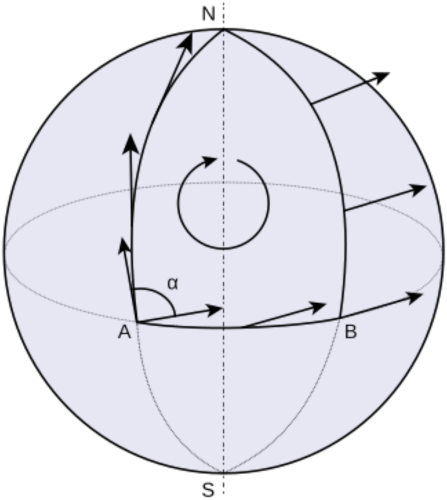

Figure: Parallel Transport

For every λ,

$$|\psi\rangle = U(t)\,|n\,(\lambda(t))\rangle$$

where $U(t)$ is time dependent phase, $|n(\lambda)\rangle$ is reference states.

Substitute into TDSE and set $E_0 = 0$,

$$U^* U = -\langle n|\,\frac{\partial}{\partial \lambda^i}\,|n\rangle\,\dot{\lambda}^i$$

Define Berry connection (Berry Potential):

$$\mathcal{A}_i(\lambda) = -i\,\langle n|\,\frac{\partial}{\partial \lambda^i}\,|n\rangle$$

Then,

$$U(t) = \exp\left(-i \int \mathcal{A}_i(\lambda)\dot{\lambda}^i dt\right)$$

Then, compute $U(t)$ along the closed path

$$e^{i\gamma_n} = \exp\left(-i \oint_C \mathcal{A}_i(\lambda) d\lambda^i\right)$$

which is the Berry phase, independent of time and depends only on the path.

$\mathcal{A}_i(\lambda)$ is a one-form over the space of parameters, and contains some gauge redundancy.
Choose

$$\left|n'(\lambda)\right\rangle = e^{i\omega}\left|n(\lambda)\right\rangle$$

then

$$\mathcal{A}'_i = \mathcal{A}_i + \frac{\partial \omega}{\partial \lambda^i}$$

exactly the same form of Gauge transformation.

We might expect the physical information in $\mathcal{A}_i$ can be found in the gauge invariant field strength, which is known as the curvature of $\mathcal{A}_i$

$$\mathcal{F}_{ij}(\lambda) = \frac{\partial \mathcal{A}_j}{\partial \lambda^i} - \frac{\partial \mathcal{A}_i}{\partial \lambda^j}$$

Then, we can rewrite Berry phase by Stokes' theorem

$$e^{i\gamma} = \exp\left(-i \oint_C \mathcal{A}_i(\lambda) d\lambda^i\right) = \exp\left(-i \int_S \mathcal{F}_{ij} dS^{ij}\right)$$

where S is two-dimensional surface in the parameter space bounded by the path C.

Example: A spin in a Magnetic Field

Consider a spin in magnetic field $\vec{B}$, with Hamiltonian

$$H = -\vec{B} \cdot \vec{\sigma} + B$$

where $\vec{\sigma}$ are the Pauli matrices.

The two eigenvalues are

$$H\left|\downarrow\right\rangle = 0 \quad \text{and} \quad H\left|\uparrow\right\rangle = 2B\left|\uparrow\right\rangle$$

where $\left|\downarrow\right\rangle$ is ground state and $\left|\uparrow\right\rangle$ is excited state. Note that, these two states are non-degenerate as long as $\vec{B} \neq 0$.

Treat the magnetic field as the parameter, $\lambda^i \equiv \vec{B}$
Express $\vec{B}$ in spherical polar coordinates with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$, the two normalised eigenstates are

$$|{\downarrow}\rangle = \begin{pmatrix} e^{i\phi} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \quad \text{and} \quad |{\uparrow}\rangle = \begin{pmatrix} e^{-i\phi} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}$$

Consider $|{\downarrow}\rangle$ state, the Berry connection are

$$\mathcal{A}_\theta = -i \langle{\downarrow}| \frac{\partial}{\partial\theta} |{\downarrow}\rangle = 0 \quad \text{and} \quad \mathcal{A}_\phi = -i \langle{\downarrow}| \frac{\partial}{\partial\phi} |{\downarrow}\rangle = -\cos^2\frac{\theta}{2}$$

The resulting Berry curvature,

$$\mathcal{F}_{\theta\phi} = \frac{\partial \mathcal{A}_\phi}{\partial \theta} - \frac{\partial \mathcal{A}_\theta}{\partial \phi} = \frac{1}{2}\sin\theta$$

Back to cartesian coordinates,

$$\mathcal{F}_{ij}(\vec{B}) = \epsilon_{ijk}\frac{B^k}{2\left|\vec{B}\right|^3}$$

which is a magnetic monopole in the space of magnetic fields!

The magnetic monopole sits at $\vec{B} = 0$ with charge $g = 1/2$. The integral of the Berry curvature over any two-dimensional sphere S^2 surrounding the origin

$$\int_{S^2} \mathcal{F}_{ij} dS^{ij} = 4\pi g = 2\pi$$

Thus, the integral of the curvature over any closed surface must be quantised in unit of 2π

$$\int \mathcal{F}_{ij} dS^{ij} = 2\pi C$$

where $C \in \mathbf{Z}$ is called the Chern number.

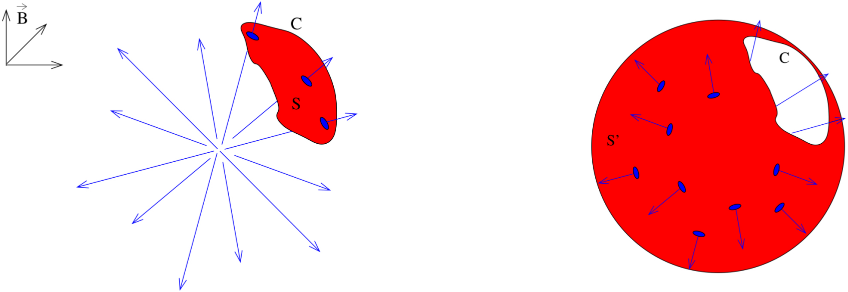

Figure: Integrating over S or S' yields the same Berry phase modulo 2π

Example: Spectral Flow

Consider a solenoid of area A, carrying magnetic field $\vec{B}$, and therefore magnetic flux $\Phi = BA$.

Outside the solenoid, $B = 0$, but vector potential exists $A_\phi = \frac{\Phi}{2\pi r}$.

Now consider a charged quantum particle lie in a ring of radius r outside the solenoid.
The energy eigenstates are

$$\psi = \frac{1}{\sqrt{2\pi r}} e^{in\phi} \quad n \in \mathbf{Z}$$

where ϕ is required to be single-valued.

Plugging into TISE, the spectrum is

$$E_n(\Phi) = \frac{\hbar^2}{2mr^2}\left(n + \frac{\Phi}{\Phi_0}\right)^2$$

where $\Phi_0 = 2\pi\hbar/e$ is quantum of flux. Note that, if $\Phi = N \times \Phi_0$,

$$E_n(N\Phi_0) = \frac{\hbar^2}{2mr^2}(n+N)^2 \quad n \in \mathbf{Z}$$

the spectrum is unaffected by the solenoid. But if $\Phi = M \times \Phi_0$

$$E_n(M\Phi_0) = \frac{\hbar^2}{2mr^2}(n+M)^2 \quad M \notin \mathbf{Z}$$

the spectrum gets shifted physically.

If $N = 1$,

$$E_n(\Phi_0) = \frac{\hbar^2}{2mr^2}(n+1)^2$$

the spectrum is still same set, but the mapping state $\rightarrow$ energy has changed. This relabeling is spectral flow.

If $M = 1/2$,

$$E_n\left(\frac{1}{2}\Phi_0\right) = \frac{\hbar^2}{2mr^2}\left(n + \frac{1}{2}\right)^2$$

the energy values themselves shift.

Thank you!