Lyapunov exponents and Jacobian Matrix

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Outline

What is Lyapunov exponent? – what does it say

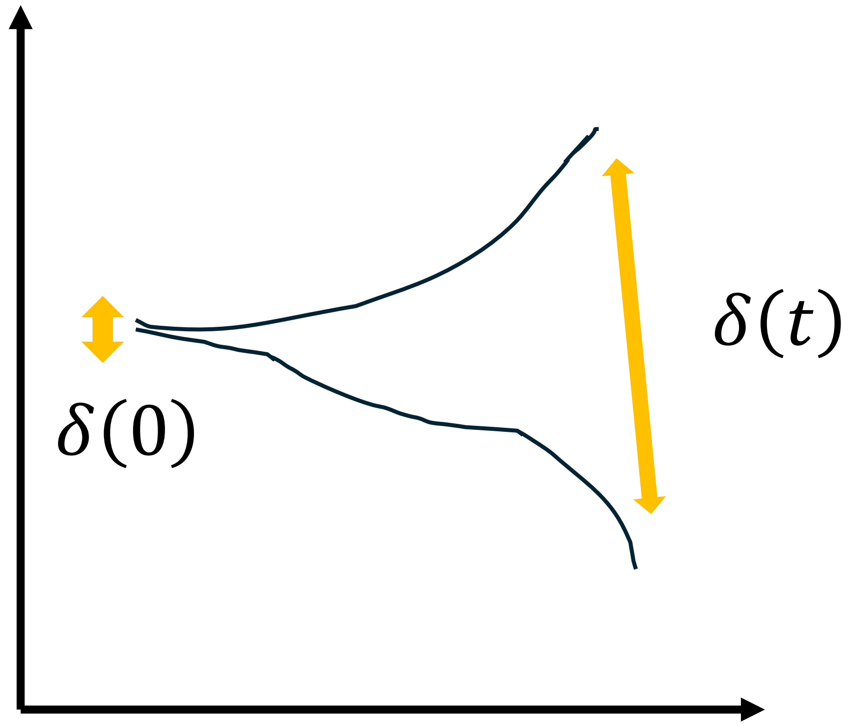
Jacobian Matrix – how a system evolves

Lyapunov exponent

Applications

What is Lyapunov exponent?

A quantity that quantify the rate of separation of infinitesimally close trajectories in a system



A way to quantify the sensitive dependence on initial condition of a system

Predictability & Chaotic behavior

Jacobian Matrix

Take 2D autonomous system for example (linearize around fix point)

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

$$\dot{(x_0 + \delta x)} = f(x_0 + \delta x, y_0 + \delta y)$$
$$\dot{(y_0 + \delta y)} = g(x_0 + \delta x, y_0 + \delta y)$$

$$\dot{(x_0 + \delta x)} = f(x_0, y_0) + \frac{\partial f(x_0, y_0)}{\partial x}\delta x + \frac{\partial f(x_0, y_0)}{\partial y}\delta y + higher\ order\ terms$$

$$\dot{(y_0 + \delta y)} = f(x_0, y_0) + \frac{\partial g(x_0, y_0)}{\partial x}\delta x + \frac{\partial g(x_0, y_0)}{\partial y}\delta y + higher\ order\ terms$$

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

$$\dot{\delta x} = \frac{\partial f(x_0, y_0)}{\partial x}\delta x + \frac{\partial f(x_0, y_0)}{\partial y}\delta y$$

$$\dot{\delta y} = \frac{\partial g(x_0, y_0)}{\partial x}\delta x + \frac{\partial g(x_0, y_0)}{\partial y}\delta y$$

$$\frac{d}{dt}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix} = \begin{bmatrix}J_{11} & J_{21}\\ J_{12} & J_{22}\end{bmatrix}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix}$$

Diagonalize Jacobian matrix

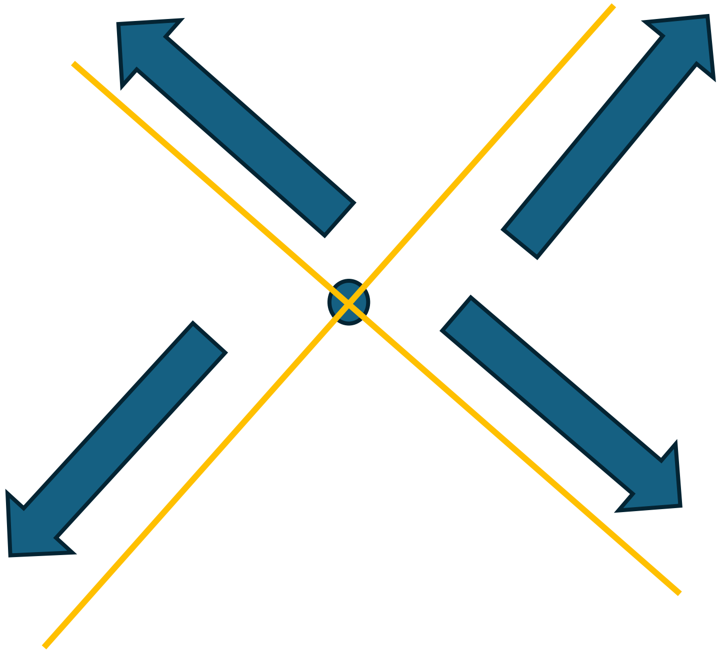
Eigenvalues

$$\begin{bmatrix}\delta x\\ \delta y\end{bmatrix} = Ae^{\lambda_1 t}\,\overrightarrow{v_1} + Be^{\lambda_2 t}\,\overrightarrow{v_2}$$

, where $\vec{v}$ stands for eigenvector of Jacobian matrix (x-y coordinate)

$$\begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = Ae^{\lambda_1 t}\,\overrightarrow{v_1} + Be^{\lambda_2 t}\,\overrightarrow{v_2}$$

$\lambda > 0$ unstable fix point



$\lambda < 0$ stable fix point

One positive one negative

Complex number

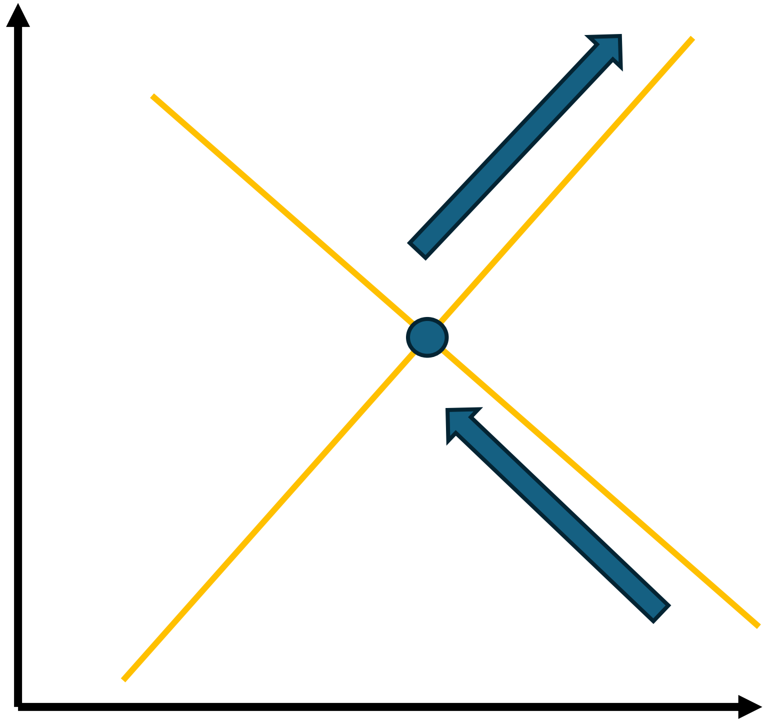
…

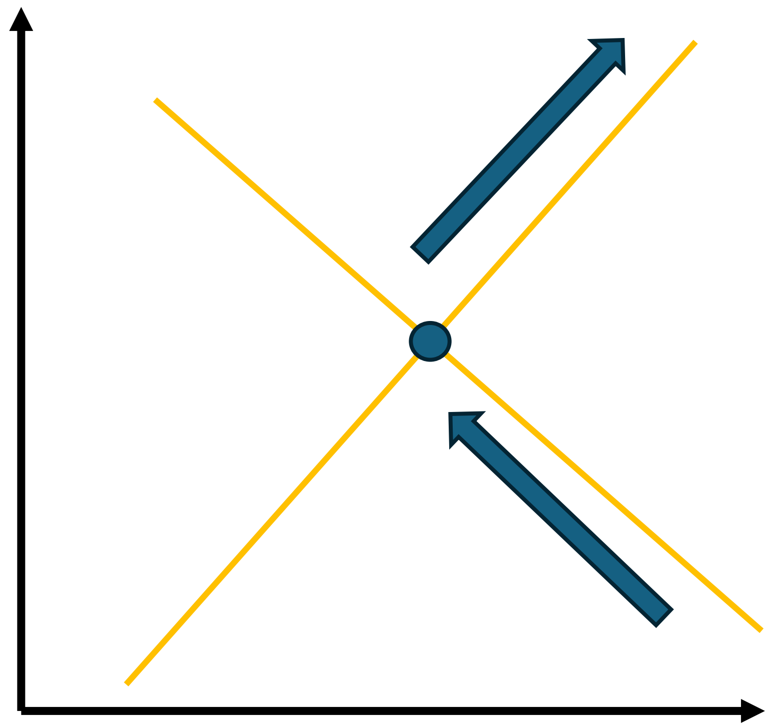
Lyapunov exponent

For n dimensional system, there are n Lyapunov exponent.
Each of them states exponential grows or shrinks along vectors in tangent space of phase space. (defined from Jacobian matrix)

$$\delta(t) \approx e^{\lambda t}\delta(0)$$

1. Not only for fix point
2. Long time average





$$\delta(t) \approx e^{\lambda t}\delta(0)$$

Positive Lyapunov exponent dominant
Chaos behavior

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

Linear approximation

$$\frac{d}{dt}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix} = \begin{bmatrix} J_{11} & J_{21} \\ J_{12} & J_{22}\end{bmatrix}\begin{bmatrix}\delta x\\ \delta y\end{bmatrix}$$

Eigenvalues

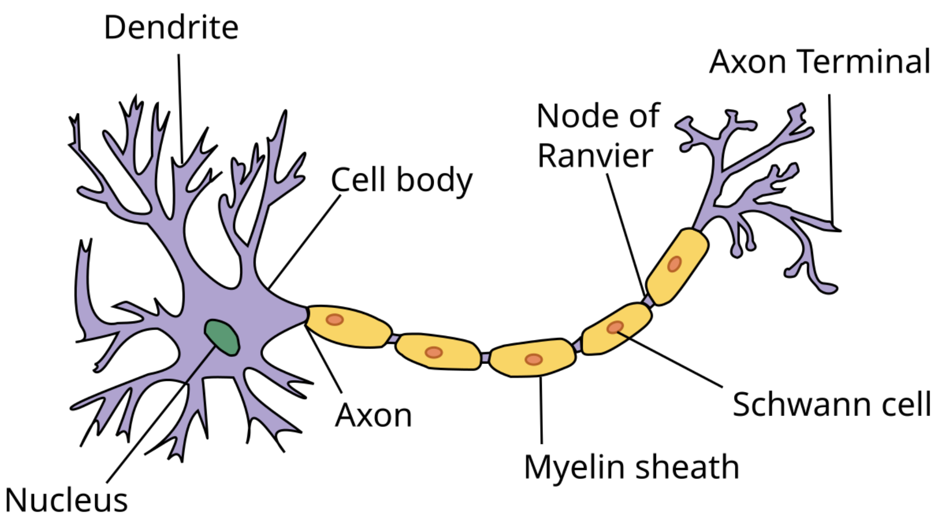
$$\begin{bmatrix}\delta x\\ \delta y\end{bmatrix} = Ae^{\lambda_1 t}\,\overrightarrow{v_1} + Be^{\lambda_2 t}\,\overrightarrow{v_2}$$

In previous example, if the system is linear, then the eigenvalues of Jacobian matrix correspond to Lyapunov exponent.

Applications in Physics

Since the concept of Jacobian matrix and Lyapunov exponent holds for dynamical equations in physical situation, it's a good tool to analysis system's behavior (stability analysis and chaotic behavior…)

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$
$$\vdots$$



Fox?

Rabbit?

In Neuron physics

1. In Hodgkin-Huxley Model, which modeling relaxation dynamics of neuron firing… , using above analyzing way helps us understand how neuron firing.

2. Stable point is crucial for our memory, that’s why we can recall things even through we didn’t recall directly on it.

3. Synaptic weight and visual balance (amblyopia)

And more….

Infectious diseases model

Outbreak of infectious disease may be related to how perturbation grows in phase space grows under some circumstance (parameter like infected rate)

How population of animal evolving with respect to environment condition after a period time (to stable state)

Stable orbit or chaotic orbit of celestial objects.

Others…

Reference

Lyapunov exponent https://en.wikipedia.org/wiki/Lyapunov_exponent

Lecture ppt : nonlinear dynamic and chaos by Kuo-An Wu

神經物理導論 Neuron Bifurcation 1 豪豬教授

https://www.youtube.com/watch?v=OqYLLRwPOhw&list=PLjT0dvu7SNXF6K1S3DhFY1a9lwNQD_Yjs&index=7

Oseledet's Theorem https://en.wikipedia.org/wiki/Oseledets_theorem

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Two weeks effort from 彭書函