<u>Helmholtz Theorem</u> (Helmholtz Decomposition)

for vector field $\vec{F}(r,t)$

if $\nabla\cdot\vec{F}(r,t) = \vec{D}(r,t)$

$\nabla\times\vec{F}(r,t) = \vec{C}(r,t)$ > are Given ($\because$ must $\nabla\cdot\vec{C}=0$) $(\forall t>0)$

then,

When $\vec{F}(r,t) \xrightarrow{|\vec{r}|\to\infty} \frac{1}{|\vec{r}|^{1+\Delta}} \to 0$ $(\Delta>0)$ [boundary condition]

haven $\vec{F}(r,t) = \frac{-1}{4\pi}\nabla\int_{all\,space} d^3r' \frac{D(\vec{r}')}{|\vec{r}-\vec{r}'|} + \frac{1}{4\pi}\nabla\times\int_{all} d^3r' \frac{\vec{C}(\vec{r}')}{|\vec{r}-\vec{r}'|}$

(uniqueness Thm)

pf for $\vec{F}(\vec{r})$, build $\vec{W}(\vec{r}) = \frac{1}{4\pi}\int_{all} d^3r' \frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}$

$\Rightarrow \nabla^2\vec{W}(\vec{r}) = \frac{1}{4\pi}\nabla^2\int_{all} d^3r' \frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|} = \frac{1}{4\pi}\int_{all} d^3r' \nabla^2\left(\frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)$

$= \frac{1}{4\pi}\int d^3r' \vec{F}(\vec{r}') = \int d^3r' (-1)\cdot 4\pi\,\delta(\vec{r}-\vec{r}')\frac{1}{4\pi}\vec{F}(\vec{r}')$

$= -\vec{F}(r)$

$\therefore \vec{F}(r) = -\nabla^2 W(r) = -\left(\nabla(\nabla\cdot\vec{W}) - \nabla\times(\nabla\times\vec{W})\right) = \nabla\times\boxed{(\nabla\times\vec{W})} - \nabla\boxed{(\nabla\cdot\vec{W})}$

$\therefore$ must. $\vec{F} = \nabla\times\vec{A} + \nabla T$ $\begin{cases} \vec{A} = \nabla\times\vec{W} \\ T = -\nabla\cdot\vec{W} \end{cases}$

$\therefore T = -\nabla\cdot\vec{W} = \frac{-1}{4\pi}\int d^3r' \nabla\cdot\left(\frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) = \frac{-1}{4\pi}\int d^3r' \vec{F}(r')\cdot\left(\nabla\frac{1}{|\vec{r}-\vec{r}'|}\right)$

$= \frac{-1}{4\pi}\int d^3r' \vec{F}(\vec{r}')\cdot(-1)\cdot\nabla'\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{4\pi}\int d^3r'\left(\nabla'\cdot\frac{\vec{F}(r')}{|\vec{r}-\vec{r}'|} - \frac{1}{|\vec{r}-\vec{r}'|}\nabla'\cdot\vec{F}(\vec{r}')\right)$

$= \underbrace{\cancel{\frac{1}{4\pi}\int_{\partial all} d\vec{a}'\cdot\frac{\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}}}_{0} - \frac{1}{4\pi}\int_{all} d^3r' \frac{\nabla'\cdot\vec{F}(r')}{|r-r'|}$

$\vec{A} = \nabla\times\vec{W} = \frac{1}{4\pi}\int_{all} d^3r' \nabla\times\left(\frac{\vec{F}(r')}{|\vec{r}-\vec{r}'|}\right) = \cdots = \frac{1}{4\pi}\int d^3r' \frac{\nabla'\times\vec{F}(\vec{r}')}{|\vec{r}-\vec{r}'|}$

application

(1)

We can look at Maxwell eq. to. show this Theorem.

$$\begin{cases} \nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0} \\ \nabla\times\vec{E} = \vec{0} \end{cases} \text{ (eletrostatics)} \qquad \begin{cases} \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{B} = \mu_0\vec{J} \end{cases} \text{ (magnetostatics)}$$

Given. Given

(2) 3D wave eq.

$$\rho\ddot{\vec{u}} = \rho\vec{f} + \mu\nabla^2\vec{u} + (\lambda+\mu)\nabla(\nabla\cdot\vec{u}) \quad \text{—(i)}$$

λ, μ: Lamé moduli (constants)

source vector

(the parameters of describing isotropy flexible solid)

$$\vec{f}(r,t) = \vec{a}\cdot g(t)\,\delta(r) = c_L^2\nabla F + c_T^2\nabla\times\vec{G}$$

$$c_L^2\nabla\vec{F} = -\nabla\cdot\left(\frac{\vec{a}}{4\pi r}\right)$$

build:

$$\vec{W}(\vec{r}) = \frac{-1}{4\pi}\int d^3r'\,\frac{\vec{a}\,g(t)\,\delta(\vec{r}')}{|\vec{r}-\vec{r}'|} = \frac{-\vec{a}\,g(t)}{4\pi}\int d^3r'\,\frac{1}{|\vec{r}-\vec{0}|} = \frac{-1}{4\pi}\vec{a}\,g(t)\,\frac{1}{r}$$

$$\vec{f} = c_L^2\nabla F + c_T^2\nabla\times\vec{G}$$

$$c_L^2 F = \nabla\cdot\left(\frac{-1}{4\pi r}\vec{a}\,g(t)\right)$$

$$c_T^2 G = -\nabla\times\left(\frac{-1}{4\pi r}\vec{a}\,g(t)\right)$$

$$\text{(i)} \rightarrow \mu\nabla^2\vec{u} + (\lambda+\mu)\nabla(\nabla\cdot\vec{u}) + \rho\left(c_L^2\nabla F + c_T^2\nabla\times\vec{G}\right) = \rho\ddot{\vec{u}}$$

$$\vec{u} = \nabla\varphi + \nabla\times\vec{\psi}$$

$$\Rightarrow \mu\nabla^2(\nabla\varphi + \nabla\times\vec{\psi}) + (\lambda+\mu)\nabla\nabla\cdot(\nabla\varphi + \nabla\times\vec{\psi}) + \rho\left(c_L^2\nabla F + c_T^2\nabla\times G\right)$$
$$= \rho(\nabla\ddot{\varphi} + \nabla\times\ddot{\vec{\psi}})$$

$$\Rightarrow \nabla\left(\nabla^2\varphi(\lambda+2\mu) + \rho c_L^2 F\right) + \nabla\times\left(\mu\nabla^2\vec{\psi} + \rho c_T^2\vec{G}\right) = \rho\nabla\ddot{\varphi} + \rho\nabla\times\ddot{\vec{\psi}}$$

$$\Rightarrow \begin{cases} (\lambda+2\mu)\nabla^2\varphi + \rho c_L^2 F = \rho\nabla\ddot{\psi} \\ \mu\nabla^2\vec{\psi} + \rho c_T^2\vec{G} = \rho\nabla\times\ddot{\vec{\psi}} \end{cases} \Rightarrow \begin{cases} \nabla^2\varphi + \frac{\rho}{\lambda+2\mu} c_L^2 F \doteq \frac{\rho}{\lambda+2\mu}\nabla\ddot{\varphi} \\ \nabla^2\vec{\psi} + \frac{\rho}{\mu} c_T^2\vec{G} = \frac{\rho}{\mu}\nabla\times\ddot{\vec{\psi}} \end{cases}$$

$$\Rightarrow \begin{cases} \nabla^2\varphi + F = \frac{1}{c_L^2}\ddot{\varphi} \\ \nabla^2\vec{\psi} + \vec{G} = \frac{1}{c_T^2}\ddot{\vec{\psi}} \end{cases}$$

$$\Rightarrow \begin{cases} \nabla^2\varphi + \frac{-1}{c_L^2}\nabla\cdot\frac{1}{4\pi r}\vec{a}\,g(t) = \frac{1}{c_L^2}\ddot{\varphi} \\ \nabla^2\vec{\psi} + \frac{1}{c_T^2}\nabla\times\frac{1}{4\pi r}\vec{a}\,g(t) = \frac{1}{c_T^2}\ddot{\vec{\psi}} \end{cases} \text{ decouple } (\checkmark)$$