Density Matrices and Entanglement Properties of the Kitaev Spin Liquid

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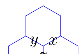
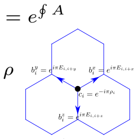
We give an exact form of the density matrix (DM) of the spin-1/2 Kitaev spin liquid represented in terms of spin operators and study the entanglement properties of the Kitaev model within the spin framework. With the explicit form of the DM, plus the exact Gauss law of the emergent gauge theory and the exact 1-form Wilson symmetry in the Kitaev model, we demonstrate the existence of the underlying block-diagonal structure of the reduced density matrix (RDM), which gives rise to the extensive degeneracy in the entanglement spectrum. The block-diagonal structure is then proven to be responsible for the separability of the entanglement entropy into the gauge and matter parts. Furthermore, the method for applying our theory to cases with an odd number of lattice sites is also discussed. It also demonstrates a relation between the entanglement spectrum and the fermion parity, which is seldom mentioned in the literature.

Introduction

- ▶ The Kitaev model $H = -\sum_{\langle ij\rangle} J_{\alpha_{ij}} \sigma_i^{\alpha_{ij}} \sigma_j^{\alpha_{ij}}$; $\alpha_{ij} = x, y, z$ .

- ▶ The Majorana-fermion representation of spins enlarges the Hilbert space: $\sigma^\mu \to ib^\mu c$. After the transformation, the Hamiltonian becomes $H = i\sum_{\langle ij\rangle} J_{\alpha_{ij}} c_i \hat{u}_{ij} c_j$, where $\hat{u}_{ij} \equiv ib_i^{\alpha_{ij}} b_j^{\alpha_{ij}}$.
- ▶ Short-range two-point correlation functions: $\langle \sigma_i^\mu \sigma_j^\nu \rangle = 0$ if (i, j) is not a pair of nearest-neighbor sites or $\mu, \nu \neq \alpha_{ij}$.
- ▶ $\mathbb{Z}_2$ gauge theory: $\hat{u}_{ij} = e^{iA_{ij}}$. The b-Majorana fermions serve as the electric field: $b_i^\mu = e^{i\pi E_{i,i+\mu}}$, $b_i^\mu \hat{u}_{i,i+\mu} b_i^\mu = i\hat{u}_{i,i+\mu} \Leftrightarrow [E_{ij}, A_{ij}] = i\delta_{ij}$.
- ▶ $\mathbb{Z}_2$ gauge flux and Stokes law: $e^{i\Phi} = W_p \equiv \prod_{\langle ij\rangle \in p} \hat{u}_{ij} = e^{\oint A}$
- ▶ The Gauss law is $b_i^x b_i^y b_i^z c_i = 1$, which reduces to $\nabla \cdot E = \rho$ if we let $c_i = e^{-i\pi\rho_i}$ be the matter field.



- ▶ Tracing out the environment to obtain an RDM $\rho_A \equiv \mathrm{Tr}_{\overline{A}}\rho$ of a region A. It is an effective local state: $\langle \mathcal{O}_A \rangle = \mathrm{Tr}\rho\mathcal{O}_A = \mathrm{Tr}_A \rho_A \mathcal{O}_A$.
- ▶ An RDM is usually expressed in a Gibbs form $\rho_A = e^{-H_A}$. The **entanglement spectrum** is the spectrum of H_A, and the **entanglement entropy** is defined by $S_A \equiv -\mathrm{Tr}_A \rho_A \ln \rho_A = \langle H_A \rangle$.

Motivation

▶How the **entanglement properties** are reflected by **spin degrees of freedom**–the true constituents of the system–remains unclear.
▶The derivation of the separability provided by [2] relies on the assumption that the subregion contains an **even number** of lattice sites, leaving the situation with an **odd number** of sites unsolved.
▶The contribution to the **entanglement from the gauge field** is not investigated as extensively as that from the fermion sector.
▶A detailed discussion of the **DM** of the Kitaev ground state, especially its explicit expression **in the spin language**, remains absent.

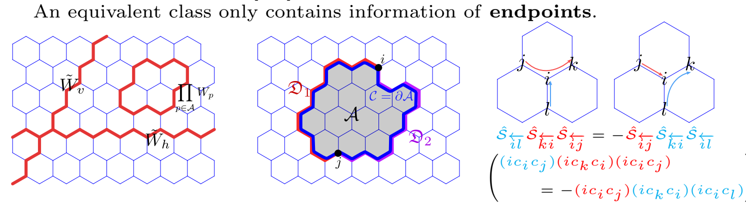
Correspondence between Strings and Fermions

- ▶ String operators and plaquette projectors as the building blocks:

$$\hat{\Sigma}_{\mathfrak{D}} \equiv \prod_{\langle ij\rangle \in \mathfrak{D}} \sigma_i^{\alpha_{ij}} \sigma_j^{\alpha_{ij}}, \quad \hat{\Pi}_P \equiv \left(\frac{\mathbb{1} + \tilde{W}_v}{2}\right)\left(\frac{\mathbb{1} + \tilde{W}_h}{2}\right)\prod_p \left(\frac{\mathbb{1} + W_p}{2}\right)$$

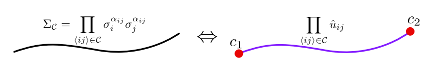
We also define $\hat{\mathcal{S}}_{\mathfrak{D}} \equiv \hat{\Pi}_P \hat{\Sigma}_{\mathfrak{D}} \hat{\Pi}_P$ as the projected string operator in the flux-free sector. The projector commutes with string operators, thus $\hat{\mathcal{S}}_{\mathfrak{D}} \equiv \hat{\Pi}_P \hat{\Sigma}_{\mathfrak{D}} \hat{\Pi}_P = \hat{\Pi}_P \hat{\Sigma}_{\mathfrak{D}}$. A similar operator $\hat{\mathcal{S}}_{\mathfrak{D}}^{(A)} \equiv \hat{\Pi}_P^{(A)} \hat{\Sigma}_{\mathfrak{D}} \hat{\Pi}_P^{(A)}$ with $\hat{\Pi}_P^{(A)} \equiv \prod_{p\in A} (\mathbb{1} + W_p)/2$ for any $\mathfrak{D} \subset A$ of a subregion A describes projected strings in A.

- ▶ Equivalence class: $\mathfrak{D} \in [\mathfrak{D}^c]$ if $\mathfrak{D} = \mathfrak{D}^c \oplus \partial\mathcal{A}$, with or without global loops. An equivalent class only contains information of **endpoints**.



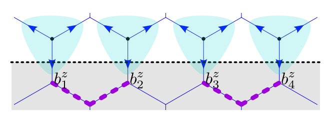
$$\hat{\mathcal{S}}_{il}\hat{\mathcal{S}}_{ki}\hat{\mathcal{S}}_{ij} = -\hat{\mathcal{S}}_{ij}\hat{\mathcal{S}}_{ki}\hat{\mathcal{S}}_{il} \quad \begin{pmatrix} (ic_ic_j)(ic_kc_i)(ic_ic_j) \\ = -(ic_ic_j)(ic_kc_i)(ic_ic_l) \end{pmatrix}$$

- ▶ Strings and the endpoints: Fermions emerge at the endpoints of string operators, and the braiding of strings reproduces the statistics [3].

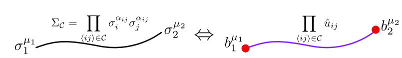


$$\hat{\Sigma}_{\mathcal{C}} \propto c_1 \left(\prod_{\langle ij\rangle\in\mathcal{C}} \hat{u}_{ij}\right) c_2$$

- ▶ Gauge-field entanglement from Gauss law [4]: Each *residual* Gauss law gives an operator that commutes with the RDM, and contributes a $\frac{1}{2}\ln 2$ to the entanglement entropy. The problem is that **the residue operators, b^μ's, are not physical**. The solution to this problem is **linking them via strings**.



$$\sigma_1^{\mu_1}\hat{\Sigma}_{\mathcal{C}}\sigma_2^{\mu_2} \propto b_1^{\mu_1}\left(\prod_{\langle ij\rangle\in\mathcal{C}} \hat{u}_{ij}\right) b_2^{\mu_2}$$



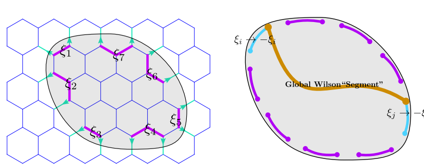
Results

▶A DM of a spin-1/2 system can be expanded by Pauli strings:

$$\rho = \frac{1}{2^N} \sum_{\{\mu_i = 0,x,y,z\}} \langle \sigma_1^{\mu_1} \cdots \sigma_N^{\mu_N} \rangle \sigma_1^{\mu_1} \cdots \sigma_N^{\mu_N}$$

▶Short-range two-point correlation: The surviving Pauli strings must be string operators $\hat{\Sigma}_{\mathfrak{D}}$.
▶DM and RDM of the Kitaev ground state:

$$\rho = \frac{1}{2^{N/2}} \sum_{[\mathfrak{D}]} \mathfrak{C}_{\mathfrak{D}} \hat{\mathcal{S}}_{\mathfrak{D}}, \quad \rho_A = \frac{1}{2^{\frac{N^A}{2} + \left(\frac{1}{2}|\partial A| - 1\right)}} \sum_{[\mathfrak{D}]\subset A} \mathfrak{C}_{\mathfrak{D}} \hat{\mathcal{S}}_{\mathfrak{D}}^{(A)}$$



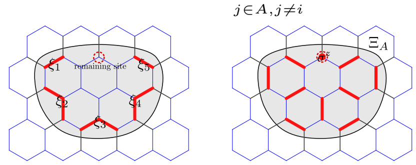
▶The residual Gauss law defines $\hat{\xi}_i \equiv \sigma_{i_1}^{\mu_{i_1}} \hat{\Sigma}_{\mathcal{C}_i} \sigma_{i_2}^{\mu_{i_2}}$, which block-diagonalize the RDM: $\rho_A = \bigoplus_{\{\xi\}} \varrho_{\{\xi\}}^{(A)}$.

▶A global Wilson operator $\hat{\mathcal{W}}$, which is a segment with endpoints in contact with the boundary, flips ξ-operators in contact with their endpoints [4]. Therefore, they relate blocks in the way that

$$\hat{\mathcal{W}} \varrho_{\{\cdots,\xi_i,\cdots,\xi_j,\cdots\}}^{(A)} \hat{\mathcal{W}} = \varrho_{\{\cdots,-\xi_i,\cdots,-\xi_j,\cdots\}}^{(A)}$$

where i and j are connected to the endpoints of $\mathcal{W}$.
▶For $N^A \in$ odd, an operator Ξ_A composed by a string operator with $\mathfrak{D}$ a dimer covering and a spin operator at the boundary pointing outwards, also commutes with the RDM, leading to an additional 2-fold degeneracy.
▶The Majorana representation of Ξ_A operator reflects its physical meaning: the **total fermion parity**

$$\Xi_A|_{\forall u=1} \propto b_i^\mu \prod_{j\in A, j\neq i} c_j$$



Outlook

1. Multiply-connected regions
2. Yao-Kivelson model [5]
3. String-gas tensor product operator [6]

References

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Conclusions

▶The DM of the Kitaev ground state expressed using spin operators is obtained, and the entanglement spectrum is thoroughly investigated in this framework.
▶The DM of the Kitaev ground state can be understood as a *superposition* of strings. The exact Wilson 1-form symmetry enables us to sort string configurations into equivalence classes according to their **endpoints**, implying that Majorana fermions emerge at the endpoints of strings. The normalization factors reflect that (i) **two** fermions emerge at the endpoints of **one** string, and (ii) the entanglement entropy contributed by the Gauss law is $S_A = \frac{1}{2}|\partial A| - 1$.
▶The ξ-operators produced by the residual Gauss law block-diagonalize the RDM, and each block is exactly the DM of the corresponding equivalent Majorana fermion system. It consists of $2^{\frac{1}{2}|\partial A|-1}$ **identical blocks** since the global Wilson operator relates all blocks with the same parity $\prod_i \xi_i$ by flipping eigenvalues of $\hat{\xi}$. The entanglement spectrum is thus that of the equivalent Majorana system with **degree of degeneracy** $2^{\frac{1}{2}|\partial A|-1}$, which contributes to the gauge-field entanglement. The term $-\ln 2$ caused by the factor 2^{-1} due to global Wilson operators is the **topological entanglement entropy**.
▶For regions with an odd number of sites, an operator Ξ_A can be found to further block-diagonalize the RDM, marking the intrinsic difference between **even and odd fermion parity**. This can be understood as the c-sector borrows a fermion from the b-sector to make the Majorana Hilbert space well-defined.