# Sturm-Liouville Theory

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# Why Do Eigenvalue Problems Matter?

Many physical systems are described by differential equations:

- Vibrating strings
- Heat conduction
- Quantum mechanics
- Wave propagation

After separation of variables, these systems often become:

**Sturm–Liouville problems**

# Sturm-Liouville Equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + [\lambda\omega(x) - q(x)]y = 0$$

Where:

- $\lambda$: eigenvalue
- $y(x)$ :eigenfunction
- $w(x)$ :weight function

Usually together with boundary conditions:

- $y(a) = y(b) = 0$

# Properties

**1. Real Eigenvalues**

$$\lambda_n \in \mathbb{R}$$

**2. Orthogonal Eigenfunctions**

$$\int_a^b y_m(x) y_n(x) w(x) dx = 0 \ (m \neq n)$$

**3. Completeness**

Any reasonable function can be expanded in terms of eigenfunctions.

# Wave Propagation in Inhomogeneous Media

Consider a vibrating string with variable density:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2}$$

Assume:

$$u(x,t) = X(x)T(t)$$

Then we obtain:

$$\frac{d}{dx}\left(T\frac{dX}{dx}\right) + \lambda\rho(x)X = 0$$

This is a Sturm–Liouville problem.

# Physical meaning

Different eigenvalues correspond to:

- Different vibration frequencies
- Different standing-wave modes

If the density changes:

- Frequencies change
- Wave patterns change

# Inverse Spectral Problem

Forward problem:

- Operator → Spectrum

Inverse problem:

- Spectrum → Operator

Goal:

- Recover properties of a system from its eigenvalues.

# Example of an Inverse Problem

Suppose we measure:

vibration frequencies of a string

Question:

Can we determine how the density varies along the string?

Different density distributions produce different spectra.

By analyzing the spectrum, we try to recover the internal structure.

Thanks for listening !