Generating Function

How it works together with special functions to shoe properties and solve problems in electromagnetics and quantum mechanics

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Definition and properties

A power series which encodes an infinite sequence as the coefficients, thus transferring a discrete problem into a continuous one.

Mathematical definition: For a sequence $\{a_n\}_{n=0}^{\infty}$, its ordinary generating function is $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

Properties: let $\mathrm{f}(x) = \sum_{n=0}^{\infty} a_n x^n$ and $\mathrm{g}(x) = \sum_{n=0}^{\infty} b_n x^n$

- **Addition:** $\mathrm{f}(x) + g(x) = \sum_{n=0}^{\infty}(a_n + b_n)x^n$.
- **Multiplication(convolution):** $f(x)g(x) = \sum_{\mathrm{n}=0}^{\infty}(\sum_{m=0}^{\infty} a_m b_{n-m})x^n$

A little example to show its mathematical use

How many sets of combinations of non negative integers x, y, z satisfy x + y + z = 20?

Set $G(x) = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$, find coefficient of x^{20} of $G(x)^3$

$G(x) = \frac{1}{1-x}, G(x)^3 = \frac{1}{(1-x)^3} = \sum_{k=0}^{\infty} \binom{k+3-1}{k} x^k$, coefficient of x^{20} is thus $\frac{22!}{20!2!} = 22 \times 21 \div 2 = 231$

Special functions in forms of generating functions

Legendre polynomials: $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \Rightarrow \Phi(t,x) = \sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\sqrt{1-2xt+t^2}}$

Hermite polynomials: $H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \Rightarrow G(t,x) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2xt-t^2}$

Application in physics 1: multipole expansion

In calculating the scalar field in electrostatics, we have the function:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r'$$

If we expand the denominator , the integral becomes complicated:

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{r}\left(1+\frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r^2}+\frac{3(\boldsymbol{r}\cdot\boldsymbol{r}')^2-r^2(r'^2)}{2r^4}+\ldots\right)$$

Let’s do this in another way by calculating the denominator:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d^3r' = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\boldsymbol{r}')}{\sqrt{r^2-2rr'cos\theta+r'^2}}d^3r' = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int \frac{\rho(\boldsymbol{r}')}{\sqrt{1-2\frac{r'}{r}cos\theta+\left(\frac{r'}{r}\right)^2}}d^3r'$$

Note that it is in the form of the generating function of the Legendre polynomials by substituting $\frac{r'}{\mathrm{r}}$ for t and $cos\theta$ for x :

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int \frac{\rho(r')}{\sqrt{1-2\frac{r'}{r}cos\theta+\left(\frac{r'}{r}\right)^2}}d^3r' = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r'^n P_n(cos\theta)\rho(r')d^3r'$$

Application in physics 1: multipole expansion

Introducing the Legendre polynomial allows us to exploit its properties to solve problems:

Recurrence : $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

Orthogonality: $\int_{-1}^{1} P_m(x)\, P_n(x)dx = \frac{2}{2n+1}\delta_{mn}$

Application in physics 2: Solving 1D quantum harmonic oscillator

This time, we start by exploring the properties of Hermite polynomials through its generating function. First, by differentiating $G(t,x)$ with respect to t , we get the recurrence relations:

$$\frac{dG(t,x)}{dt} = 2(x-t)e^{2xt-t^2} = 2x\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} - 2\sum_{n=0}^{\infty} H_n(x)\frac{t^{n+1}}{n!}$$

And the left-hand side:

$$\frac{dG(t,x)}{dt} = \frac{d}{dt}\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!}$$

Compare the terms with the same order, we get:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

Application in physics 2: Solving 1D quantum harmonic oscillator

Now, let’s attempt to multiply the generating function with itself:

$$G(t,x)G(s,x) = e^{2x(t+2)-t^2-s^2}$$

Integrating with a gaussian weight function e^{-x^2}, we get:

$$\int_{-\infty}^{\infty} e^{2x(t+2)-t^2-s^2} e^{-x^2}\,dx = \sqrt{\pi}e^{2ts} = \sqrt{\pi}\sum_{k=0}^{\infty} \frac{2^k (ts)^k}{k!} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx\right)\frac{t^m s^n}{m!\,n!}$$

The last equal sign suggests that $k = m = n$, indicates that all terms of $m \neq n$ vanishes. Thus, we discovers the orthogonality of Hermite polynomials:

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx = \sqrt{\pi}2^n n!\,\delta_{mn}$$

Application in physics 2: Solving 1D quantum harmonic oscillator

By algebraic conduction, we know that the wave function to the 1D quantum harmonic oscillator is:

$$\psi_n(x) = N_n H_n(\alpha x) e^{-\frac{1}{2}\alpha^2 x^2}$$

By the recurrence relation, we can rapidly know the wave function of the next order. By the orthogonality, we can calculate the matrix element with much less work.