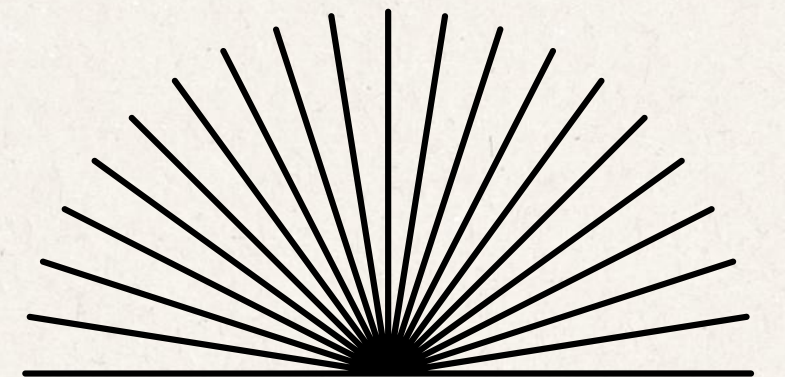


# Generating Function

How it works together with special functions to show properties and solve problems in electromagnetics and quantum mechanics

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# Definition and properties

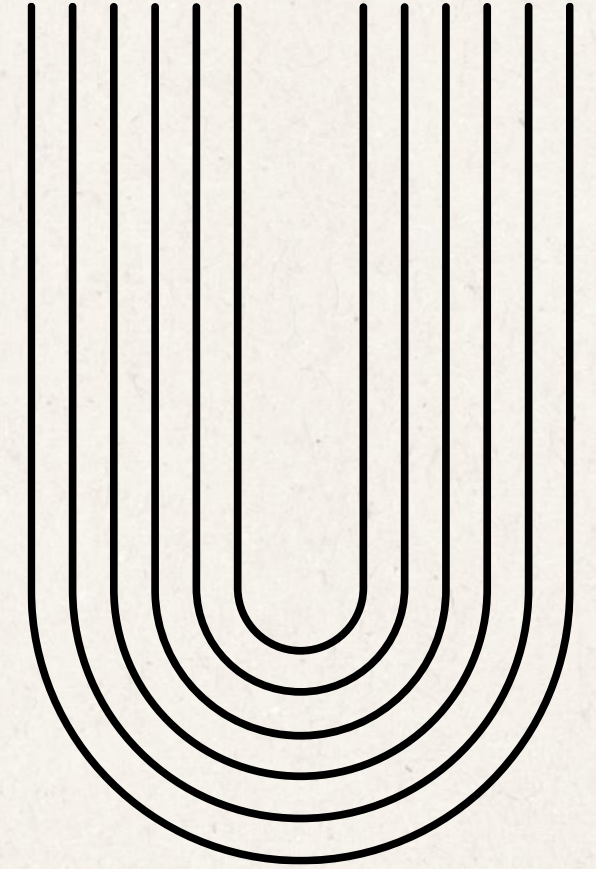
A power series which encodes an infinite sequence as the coefficients, thus transferring a discrete problem into a continuous one.

**Mathematical definition:** For a sequence  $\{a_n\}_{n=0}^{\infty}$ , its ordinary generating function is  $G(x) = \sum_{n=0}^{\infty} a_n x^n$ .

**Properties:** let  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  and  $g(x) = \sum_{n=0}^{\infty} b_n x^n$

**Addition:**  $f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)x^n$ .

**Multiplication(convolution):**  $f(x)g(x) = \sum_{n=0}^{\infty} (\sum_{m=0}^{\infty} a_m b_{n-m})x^n$



# A little example to show its mathematical use

How many sets of combinations of non negative integers  $x, y, z$  satisfy  $x + y + z = 20$ ?

Set  $G(x) = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$ , find coefficient of  $x^{20}$  of  $G(x)^3$

$G(x) = \frac{1}{1-x}$ ,  $G(x)^3 = \frac{1}{(1-x)^3} = \sum_{k=0}^{\infty} \binom{k+3-1}{k} x^k$ , coefficient of  $x^{20}$  is thus  $\frac{22!}{20!2!} = 22 \times 21 \div 2 = 231$

# Special functions in forms of generating functions

Legendre polynomials:  $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \Rightarrow \Phi(t, x) = \sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{\sqrt{1-2xt+t^2}}$

Hermite polynomials:  $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \Rightarrow G(t, x) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2}$

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# Application in physics 1: multipole expansion

In calculating the scalar field in electrostatics, we have the function:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'$$

If we expand the denominator, the integral becomes complicated:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} \left( 1 + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} + \frac{3(\mathbf{r} \cdot \mathbf{r}')^2 - r^2(r'^2)}{2r^4} + \dots \right)$$

Let's do this in another way by calculating the denominator:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\sqrt{r^2 - 2rr' \cos\theta + r'^2}} d^3r' = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \int \frac{\rho(\mathbf{r}')}{\sqrt{1 - 2\frac{r'}{r} \cos\theta + \left(\frac{r'}{r}\right)^2}} d^3r'$$

Note that it is in the form of the generating function of the Legendre polynomials by substituting  $\frac{r'}{r}$  for  $t$  and  $\cos\theta$  for  $x$ :

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \int \frac{\rho(\mathbf{r}')}{\sqrt{1 - 2\frac{r'}{r} \cos\theta + \left(\frac{r'}{r}\right)^2}} d^3r' = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int r'^n P_n(\cos\theta) \rho(\mathbf{r}') d^3r'$$

# Application in physics 1: multipole expansion

Introducing the Legendre polynomial allows us to exploit its properties to solve problems:

$$\text{Recurrence : } (n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x)$$

$$\text{Orthogonality: } \int_{-1}^1 P_m(x) P_n(x) dx = \frac{2}{2n+1} \delta_{mn}$$

# Application in physics 2: Solving 1D quantum harmonic oscillator

This time, we start by exploring the properties of Hermite polynomials through its generating function. First, by differentiating  $G(t, x)$  with respect to  $t$ , we get the recurrence relations:

$$\frac{dG(t, x)}{dt} = 2(x - t)e^{2xt - t^2} = 2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} - 2 \sum_{n=0}^{\infty} H_n(x) \frac{t^{n+1}}{n!}$$

And the left-hand side:

$$\frac{dG(t, x)}{dt} = \frac{d}{dt} \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} H_n(x) \frac{t^{n-1}}{(n-1)!}$$

Compare the terms with the same order, we get:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

# Application in physics 2: Solving 1D quantum harmonic oscillator

Now, let's attempt to multiply the generating function with itself:

$$G(t, x)G(s, x) = e^{2x(t+s)-t^2-s^2}$$

Integrating with a gaussian weight function  $e^{-x^2}$ , we get:

$$\int_{-\infty}^{\infty} e^{2x(t+s)-t^2-s^2} e^{-x^2} dx = \sqrt{\pi} e^{2ts} = \sqrt{\pi} \sum_{k=0}^{\infty} \frac{2^k (ts)^k}{k!} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx \right) \frac{t^m s^n}{m! n!}$$

The last equal sign suggests that  $k = m = n$ , indicates that all terms of  $m \neq n$  vanishes. Thus, we discovers the orthogonality of Hermite polynomials:

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = \sqrt{\pi} 2^n n! \delta_{mn}$$

# Application in physics 2: Solving 1D quantum harmonic oscillator

By algebraic conduction, we know that the wave function to the 1D quantum harmonic oscillator is:

$$\psi_n(x) = N_n H_n(\alpha x) e^{-\frac{1}{2}\alpha^2 x^2}$$

By the recurrence relation, we can rapidly know the wave function of the next order. By the orthogonality, we can calculate the matrix element with much less work.