Legendre Transformation

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Why is Legendre transformation important

Sometimes the natural variables are not the most useful ones.

Legendre transform changes the description:

$\dot{q} \rightarrow p$ Lagrangian → Hamiltonian mechanics

$S \rightarrow T$ Internal energy → Helmholtz free energy

Definition

$$p = \frac{df}{dx}$$

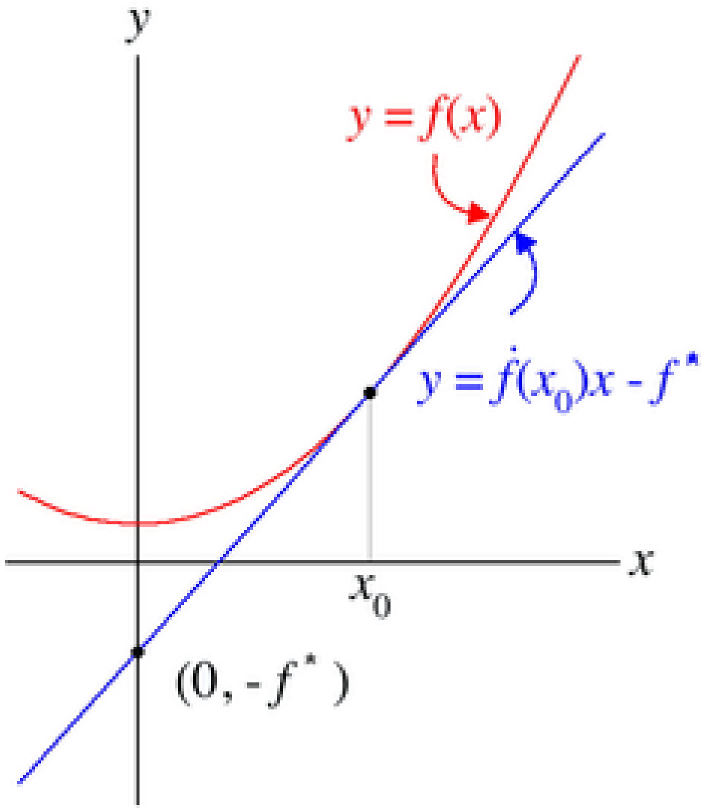
p : conjugate variable of x

$$f(x) = px - g(p)$$

$$g(p) = px - f(x)$$
$$with\ x = x(p)$$

Legendre transform replaces a variable by its conjugate variable.

Geometric structure

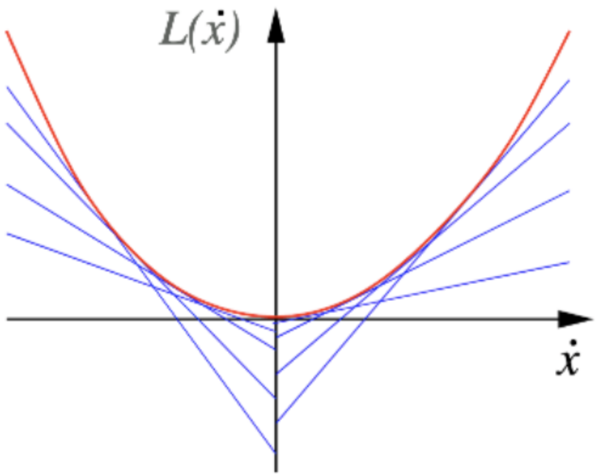


$p = f'(x_0)$

$f^*(p) = p\,x_0 - f(x_0)$

- f(x): described by the position x_0
- f*(p): described by the tangent slope p
- The same curve is encoded by its family of tangent lines

point description → slope description



Simple example

$$f(x) = \frac{1}{2}kx^2 \xrightarrow{\text{Legendre transform}} f^*(p) = \frac{p^2}{2k}$$

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$$p = \frac{df}{dx} = kx$$

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$$f^*(p) = px - f(x) = p\left(\frac{p}{k}\right) - \frac{1}{2}kx^2 = \frac{p^2}{2k}$$

Application I: Classical mechanics

Lagrangian description:

$L = L(q, \dot{q})$

Conjugate momentum:

$$p \equiv \frac{\partial L}{\partial \dot{q}}$$

Legendre transform:

$H(q, p) = p\dot{q} - L(q, \dot{q})$

Example: Free particle

$$L = \frac{1}{2}m\dot{q}^2 - V(q)$$

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \qquad \dot{q} = \frac{p}{m}$$

$$H = p\dot{q} - L$$

$$H = \frac{p^2}{2m} + V(q)$$

Application II: Thermodynamics

Internal energy:

$E = E(S,V)$

$dE = TdS - PdV$

Conjugate variables:

$$T = \left(\frac{\partial E}{\partial S}\right)_V \qquad S \leftrightarrow T$$

$$-P = \left(\frac{\partial E}{\partial V}\right)_S \qquad V \leftrightarrow -P$$

Legendre transform:

$$F = E - TS$$

$$dF = dE - TdS - SdT$$

$$dE = TdS - PdV$$

$$dF = -S\,dT - P\,dV$$

$$\rightarrow F = F(T,V)$$

Conclusion

$$x \to p = \frac{df}{dx}$$

Legendre transform is not just algebra.

It replaces a variable by its conjugate variable.

$$\dot{q} \to p = \frac{\partial L}{\partial \dot{q}}$$

It connects different but equivalent descriptions of the same system.

$$S \to T = \left(\frac{\partial E}{\partial S}\right)_V$$

Thanks for listening