



Introduction.

Imagine you have a furry ball, and you try to comb it down.

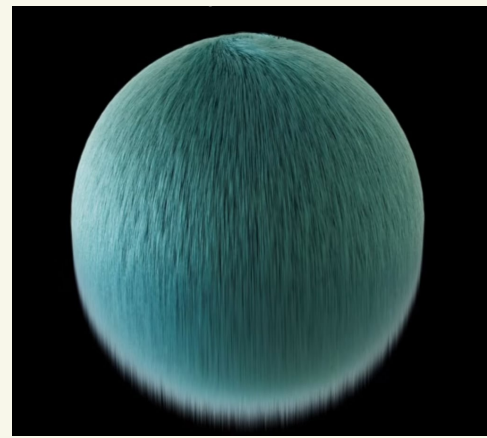
Is there a way to do it without having the fur stick up at at least one point?

For example, you try to comb it counterclockwise around some arbitrary axis.

However, it ends up with two swirls at the top and the bottom. The hair at the centerpoint of those swirls would have nowhere to go but forced to stick up.

Well, no matter how hard you try, there's just no way you can find such condition. This is a mathematical guarantee that you will be left with at least one tuft like this.

This is what I am going to introduce today.



The Hairy Ball Theorem.

Hairy Ball Thm:

Any continuous vector field $v: S^2 \rightarrow \mathbb{R}^3$ that is tangent to the sphere must have at least one point that vanishes.

2-sphere maps to 3-D space

↓

for every point on a sphere, this function assigns a vector to it.

Euler Characteristic (歐拉示性數)

Basic Def: It is a "Topological Invariant", which means no matter how an object was stretched, bent, or squeezed (as long as it's not torn apart or glued together), this value stays constant.

Symbol: χ (pronounced as "chi")

Formula: $\chi = (\text{Vertices}) - (\text{Edges}) + (\text{Faces})$
頂點數 邊數 面數

Examples: For sphere, cube, pyramids

$$\Rightarrow \chi = 2$$

Why? Let's calculate the χ of a cube:

$$\chi = 8 - 12 + 6 = 2$$

And since cubes, spheres, and pyramids can be turned into each other by squeezing, their Euler characteristic are also "2".

$\Rightarrow \chi$ of a sphere equals 2.

\Rightarrow We can also imply that any curved surface without any hole has a χ of 2.

\hookrightarrow Keep this in mind!

Poincaré-Hopf Theorem (龐加萊-霍普夫定理)

\hookrightarrow meaning M looks like a flat space.

(i.e. Euclidean space) locally.

Let M be a compact, orientable, differentiable manifold. Let V be a continuous vector field on M with a finite number of isolated zeros $p_1, p_2, p_3, \dots, p_k$. Then the sum of the indices of V at these zeros is equal to the Euler characteristic of M :

\hookrightarrow 封閉 & 有正反面之分.

\hookrightarrow smooth

vector field on M with a finite number of isolated zeros $p_1, p_2, p_3, \dots, p_k$. Then the sum of the indices of V at these zeros is equal to the Euler characteristic of M :

$$\sum_{i=1}^k \text{index}_V(p_i) = \chi(M)$$

add up all the index from different indices

zero

in some sort of vector field

Right Hand Side: Euler characteristic of a manifold M .




Left Hand Side:

What's "index"?

Imagine a little ant walk around a zero point. Index equals how many full circles the vector field beneath it has turned.

We have a formula for this:

$$\text{Index} = \frac{1}{2\pi} \times \text{change in the angle } (\Delta\alpha)$$

Ex: " for "  "sources",  "sink", and  "center" ←

since the vector spin a full circle

$$\Rightarrow \Delta \text{ increased by } 2\pi$$

$$\Rightarrow \bar{\text{index}} = \frac{1}{2\pi} \times 2\pi = +1$$

These are all kinds of zeros

2. for "saddle point" ←

since the angle decreased instead,

$$\Rightarrow \Delta\alpha = -2\pi$$

$$\Rightarrow \bar{\text{index}} = \frac{1}{2\pi} \times (-2\pi) = -1$$

3. if $\bar{\text{index}} = 0 \Rightarrow$ all vectors are parallel to each other

\Rightarrow no zero there at all

Recall that the χ of a sphere is 2

$$\Rightarrow \sum_i \bar{\text{index}}_v(p_i) = 2 \text{ when it comes to a sphere}$$

Therefore, it's impossible that there isn't any zero on the surface of a sphere.

\Rightarrow Hairy Ball Theorem Successfully Derived!

* Notification: $\sum_i \bar{\text{index}}_v(p_i) = 2$ does not imply that there should always be 2 swirls. It only tells that the sum of all indices should always be 2. Therefore, cases such as 4 swirls plus 1 saddle point also fulfill this rule.

$$\hookrightarrow 4 \times (+1) + 1 \times (-2) = 2 \quad (\checkmark)$$

Exemplifications:

① Atmospheric Circulation:

How come there is always an cyclone in Earth's wind field?

\Rightarrow { Earth itself is a differentiable manifold M .

Wind pointing in different directions can be seen as a vector field \vec{V} .

According to HBT \Rightarrow There **must** exist at least 1 zero that makes $\sum_i \bar{\text{index}}_v(p_i) = 2$

\Rightarrow It is also obvious that there are countless combinations that can configure an index sum of 2.

② Nuclear Fusion (核融合)

We've discussed that there ought to be zeros on a sphere; therefore, when designing the reactor for nuclear fusion, we would never take spherical shape into consideration, otherwise the plasma will shoot out at some point(s) and cause grave damage to everything.

What should we do then?

⇒ Introduce: Taurus 

For a taurus: $\chi = "0"$  $\left(\begin{array}{l} 8-12+4 \\ = 0 \end{array} \right)$

↳ so $\sum_i \text{index}_v(p_i) = 0 \Rightarrow$ exists a distribution that contains 0 zero
There vector
⇒ no plasma will shoot outward

⇒ A safe design for a nuclear fusion reactor.

