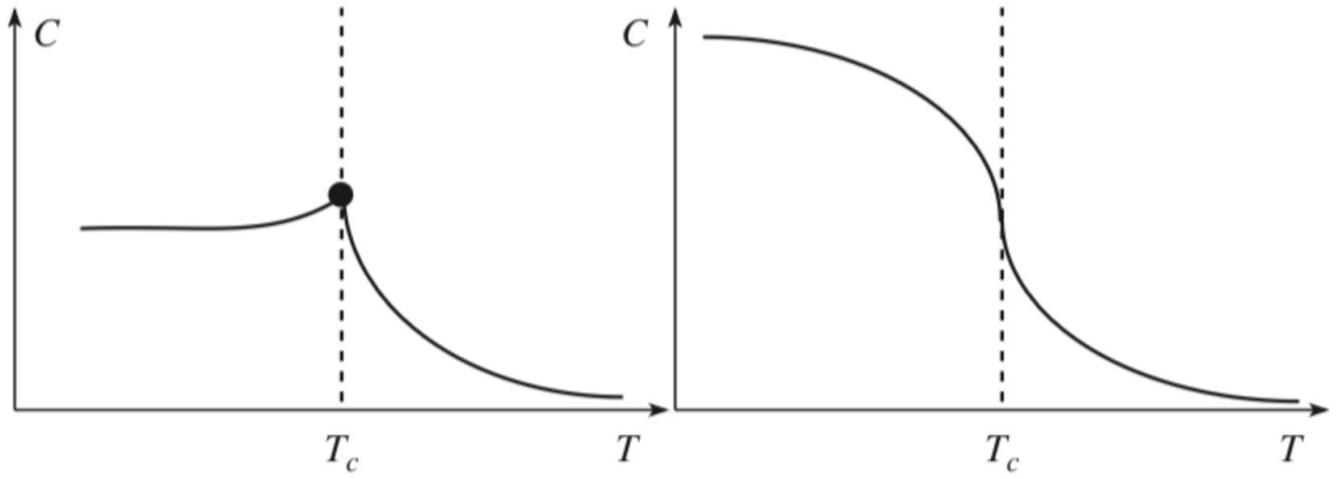
Final Presentation of Mathematical Physics

Renormalization Group Procedure in Statistical Field Theory

Introduction

- A central problem in early condensed matter physics is the theory of phase transition of various substances, e.g. ferromagnetism-paramagnetism transition and liquid-solid transition. At transition, the divergence of thermal-dynamical quantities are characterized by *critical exponents*.



$$C_{\pm}(T, h = 0) \propto |t|^{-\alpha_{\pm}}.$$

Early attempts-Landau Theory

- L. Landau and V. Ginzburg formulated their mean-field approach by constructing an effective Hamiltonian by focusing on the *symmetry* of the system. This approach qualitatively captures the phase transition.
- The essence is the mean-field solution that exploits the saddle-point approximation in the evaluation of the partition function.

$$Z = \int D[m(\vec{x})] \exp\left(-\beta H\left(m(\vec{x})\right)\right) \cong \exp(-\beta H\left(m(\vec{x})\right)_{min})$$

In the above equation $\int D[m(\vec{x})]$ means *"summing all possible field configrations"* .

Early attempts-Landau Theory

- The effective Landau-Ginzburg Hamiltonian (ferromagnetism) is

$$\beta H[m(\vec{x})] = \int d\vec{x}^{d}\left(\frac{t}{2}\vec{m}^{2} + \frac{K}{2}(\nabla m)^{2} + u\vec{m}^{4} + \cdots - \vec{h}\cdot\vec{m}\right)$$

The above equation assumes *space-translational* and *rotational symmetry*, that's why we ruled out all first order dependence and only include terms that are rotationally invariant.

To keep the energy minimum, we shall set K=0 and result in:

$$Z \approx Z_{\mathrm{sp}} = \mathrm{e}^{-\beta F_0}\int \mathrm{d}\vec{m}\exp\left[-V\left(\frac{t}{2}m^{2} + um^{4} + \cdots - \vec{h}.\vec{m}\right)\right]$$

Early attempts-Landau Theory

- Partition function:
- $Z = \int D[m(\vec{x})] \exp\left(-\beta H\left(m(\vec{x})\right)\right) = \int D[m(\vec{x})] \exp\left(\int d\vec{x}^d (\frac{t}{2}\vec{m}^2 + \frac{K}{2}(\nabla m)^2 + u\vec{m}^4 + \cdots - \vec{h}\cdot\vec{m})\right)$
- We can employ saddle-point approximation to evaluate the above "functional integral":

$$Z \approx Z_{\mathrm{sp}} = \mathrm{e}^{-\beta F_0} \int \mathrm{d}\vec{m} \exp\left[-V\left(\frac{t}{2}m^2 + um^4 + \cdots - \vec{h}.\vec{m}\right)\right]$$

$$\beta F_{\mathrm{sp}} = -\ln Z_{\mathrm{sp}} \approx \beta F_0 + V \min\{\Psi(\vec{m})\}_{\vec{m}},$$

- The minimum configuration $\min\{\Psi(\vec{m})\}$ can be found as:

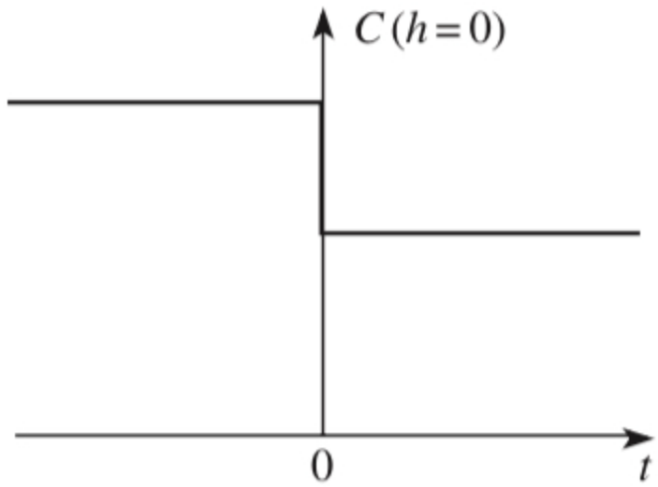
$$\Psi'(\overline{m}) = t\overline{m} + 4u\overline{m}^3 + \cdots - h = 0.$$

Critical behavior

- The behavior of heat capacity at critical point can be calculated as:

$$C(h=0) = -T\frac{\partial^2 F}{\partial T^2} \approx -T_c a^2 \frac{\partial^2}{\partial t^2}(k_B T_c \beta F) = C_0 + V k_B a^2 T_c^2 \times \begin{cases} 0 & \text{for } t > 0, \\ \dfrac{1}{8u} & \text{for } t < 0. \end{cases}$$

- The mean-field theory predicts a step-Function behavior.



Comparison with experiment

- We can compare the calculation to the measured values:

Transition type	Material	α	β	γ	ν
Ferromagnets ($n = 3$)	Fe, Ni	-0.1	0.4	1.3	
Superfluid ($n = 2$)	He^4	0	0.3	1.3	0.7
Liquid–gas ($n = 1$)	CO_2, Xe	0.1	0.3	1.2	0.7
Ferroelectrics and superconductors	TGS	0	1/2	1	1/2
Mean-field theory		0	1/2	1	1/2

Fluctuation from the mean-field solution

- The failure of mean-field solution suggests that at critical point, the assumption that the field configuration is uniform is invalid. We need to examine the effect of *fluctuation*, there is a correction to the heat-capacity:

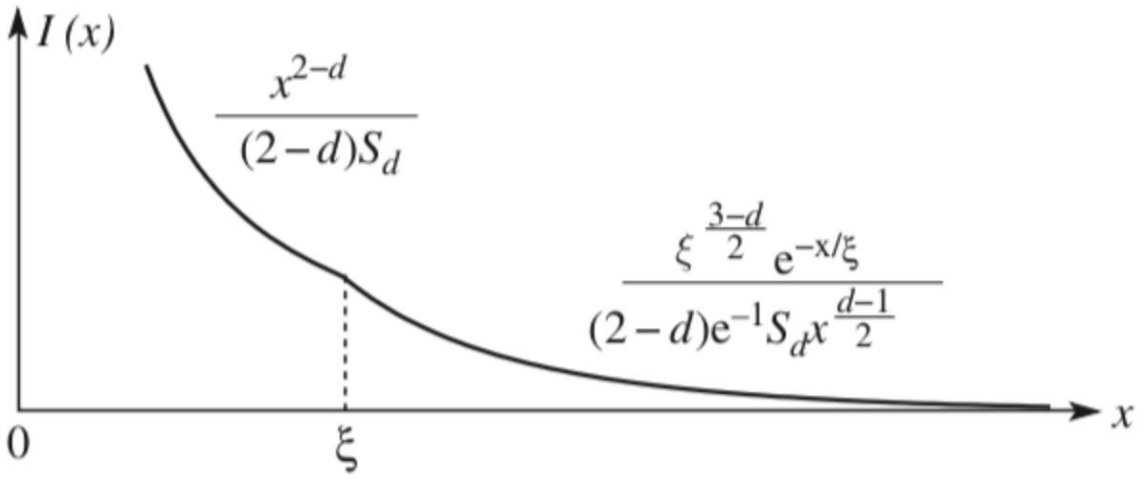
$$C_{\text{singular}} \propto -\frac{\partial^2(\beta f)}{\partial^2 t} = \begin{cases} 0 + \frac{n}{2}\int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\frac{1}{(Kq^2+t)^2} & \text{for } t>0 \\ \frac{1}{8u} + 2\int \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\frac{1}{(Kq^2-2t)^2} & \text{for } t<0. \end{cases}$$

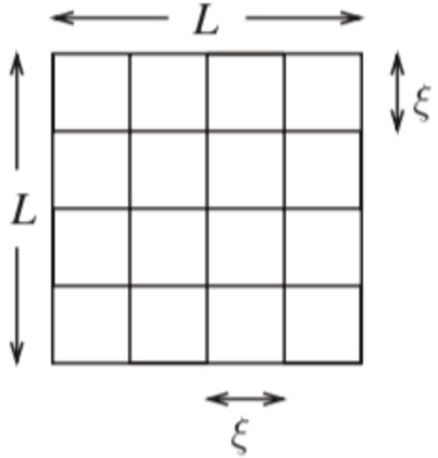
Which can be approximated as:

$$C_F \simeq \frac{1}{K^2}\begin{cases} a^{4-d} & \text{for } d>4 \\ \xi^{4-d} & \text{for } d<4. \end{cases}$$

Correlation function

- The correlation function of the system behaves as *monomial* within a length scale, and *decays exponentially* outside this scale. This scale is called correlation length ξ.
- The monomial correlation in the length scale ξ implies the system is *scale invariant* and is associated with *self-similarity*.





RG Procedures and self-similarity

- The correlation function at critical point implies that the fluctuation of the system are *scale-invariant*. L.Kadanoff suggested an approach that redefines the statistical field in the self-similar domain. It is reasonable to do this due to self-similarity.

RG Procedures and self-similarity

We can thus summarize the RG process as:

(1)Coarse-grain:

For a positive number b, average the field in each block with length ab

(2)Rescale:

Set the scale to $\frac{x}{b}$

(3)Renormalize:

Set the field to $\acute{m}(x) = \zeta m(x)$

After this process, we will still get a self-similar field configuration if the system is scale invariant. Critical points thus correspond to *fixed points* of RG procedure.

RG process and RG flow

- The RG procedure introduced above is equivalent to a redefinition of the structure constants $(t, K, u, h, \ldots.)$ to another set of variables $(t', K', u', h' \ldots\ldots)$. The reason is that before and after RG procedure, the resulting field are constraint by the same *symmetry*.
- The RG procedure can thus be written as:

$$\begin{cases} t_b(t,h) = A(b)t + \cdots \\ h_b(t,h) = D(b)h + \cdots \end{cases}$$

Note that the effect of RG operation of scale b_1 then b_1 is equivalent to the operation of scale $b_1 b_2$, so we can write $A(b) = b^{y_t}$ and $\mathrm{D}(b) = b^{y_h}$

RG process and RG flow

- After each RG step, the various parameters scale as:

$$\begin{cases} t' \equiv t_b = b^{y_t} t + \cdots \\ h' \equiv h_b = b^{y_h} h + \cdots \end{cases}$$

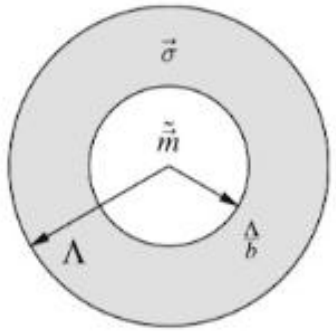
The parameters with exponents less than one will scale to zero. This explains the *universality* of phase transition, as many parameters are not important near critical point.

Applications

- 1. Modification of Structural Constants:

Wilson's perturbative approach:

(1) Coarse-graining: This is most conveniently done in Fourier basis. The averaging over short length scale in real-space is equivalent to averaging over large momentum in Fourier space.



$$\vec{m}(\mathbf{q}) = \begin{cases} \tilde{\vec{m}}(\mathbf{q}) & \text{for } 0 < q < \Lambda/b \\ \vec{\sigma}(\mathbf{q}) & \text{for } \Lambda/b < q < \Lambda. \end{cases}$$

Applications

The partition function in Fourier space can be evaluated as

$$Z = \int \mathcal{D}\tilde{\vec{m}}(\mathbf{q})\mathcal{D}\vec{\sigma}(\mathbf{q})\exp\left\{-\int_0^{\Lambda}\frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\left(\frac{t+Kq^2}{2}\right)\right.$$
$$\left.(|\tilde{m}(\mathbf{q})|^2+|\sigma(\mathbf{q})|^2)-\mathcal{U}[\tilde{\vec{m}}(\mathbf{q}),\vec{\sigma}(\mathbf{q})]\right\},$$

By dividing the integration domain into long and short-wavelength region. Coarse-graining amounts to integrating over large-momentum region:

$$Z = \int \mathcal{D}\tilde{\vec{m}}(\mathbf{q})\exp\left\{-\int_0^{\Lambda/b}\frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\left(\frac{t+Kq^2}{2}\right)|\tilde{m}(\mathbf{q})|^2\right\}$$
$$\times\exp\left\{-\frac{nV}{2}\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\ln\left(t+Kq^2\right)\right\}\left\langle \mathrm{e}^{-\mathcal{U}[\tilde{\vec{m}},\vec{\sigma}]}\right\rangle_{\sigma}$$
$$\equiv\int\mathcal{D}\tilde{\vec{m}}(\mathbf{q})\mathrm{e}^{-\beta\tilde{\mathcal{H}}[\tilde{\vec{m}}]},$$

where

$$\langle\mathcal{O}\rangle_{\sigma}\equiv\int\frac{\mathcal{D}\vec{\sigma}(\mathbf{q})}{Z_{\sigma}}\mathcal{O}\exp\left[-\int_{\Lambda/b}^{\Lambda}\frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\left(\frac{t+Kq^2}{2}\right)|\sigma(\mathbf{q})|^2\right],$$

Applications

- The coarse grained Hamiltonian is thus:

$$\beta\tilde{\mathcal{H}}[\tilde{\bar{m}}] = V\delta f_b^0 + \int_0^{\Lambda/b} \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\left(\frac{t+Kq^2}{2}\right)|\tilde{m}(\mathbf{q})|^2 - \ln\left\langle \mathrm{e}^{-\mathcal{U}[\tilde{\bar{m}},\bar{\sigma}]}\right\rangle_\sigma .$$

- We can expand the last term by cumulant expansion:

$$\ln\left\langle \mathrm{e}^{-\mathcal{U}}\right\rangle_\sigma = -\langle\mathcal{U}\rangle_\sigma + \frac{1}{2}\left(\langle\mathcal{U}^2\rangle_\sigma - \langle\mathcal{U}\rangle_\sigma^2\right) + \cdots + \frac{(-1)^\ell}{\ell!} \times \ell\text{th cumulant of } \mathcal{U} + \cdots$$

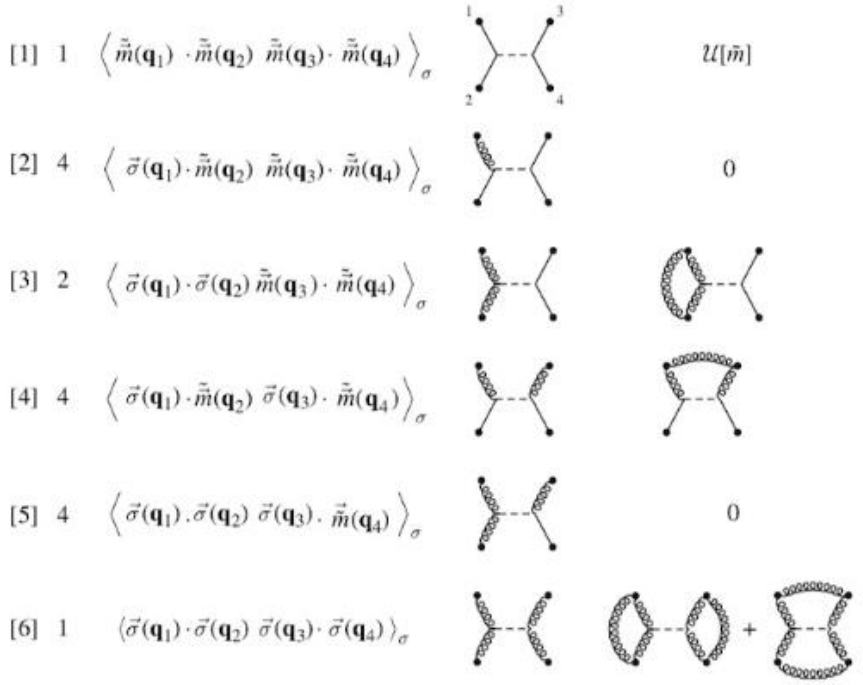
First, we shall only keep the first order.

Applications

- The first order correction to quadratic model is

$$\left\langle \mathcal{U}\left[\tilde{\vec{m}},\vec{\sigma}\right]\right\rangle_{\sigma} = u\int \frac{\mathrm{d}^d\mathbf{q}_1\mathrm{d}^d\mathbf{q}_2\mathrm{d}^d\mathbf{q}_3\mathrm{d}^d\mathbf{q}_4}{(2\pi)^{4d}}(2\pi)^d\delta^d(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3+\mathbf{q}_4)$$
$$\left\langle\left[\tilde{\vec{m}}(\mathbf{q}_1)+\vec{\sigma}(\mathbf{q}_1)\right]\cdot\left[\tilde{\vec{m}}(\mathbf{q}_2)+\vec{\sigma}(\mathbf{q}_2)\right]\right.$$
$$\left.\times\left[\tilde{\vec{m}}(\mathbf{q}_3)+\vec{\sigma}(\mathbf{q}_3)\right]\cdot\left[\tilde{\vec{m}}(\mathbf{q}_4)+\vec{\sigma}(\mathbf{q}_4)\right]\right\rangle_{\sigma}.$$

The calculation can be aided by graphical methods, These graphs help us identify which elements are zero, so we don't have to go through the integrals and then find out it doesn't contribute.



Applications

- The results of the above calculation is

$$\beta\tilde{\mathcal{H}}[\tilde{\bar{m}}] = V\left(\delta f_b^0 + u\delta f_b^1\right) + \int_0^{\Lambda/b} \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\left(\frac{\tilde{t}+Kq^2}{2}\right)|\tilde{m}(\mathbf{q})|^2$$
$$+ u\int_0^{\Lambda/b} \frac{\mathrm{d}^d\mathbf{q}_1\mathrm{d}^d\mathbf{q}_2\mathrm{d}^d\mathbf{q}_3}{(2\pi)^{3d}}\tilde{\bar{m}}(\mathbf{q}_1)\cdot\tilde{\bar{m}}(\mathbf{q}_2)\tilde{\bar{m}}(\mathbf{q}_3)\cdot \quad \tilde{K}=K, \quad \text{and} \quad \tilde{u}=u. \quad),$$

where

$$\tilde{t} = t + 4u(n+2)\int_{\Lambda/b}^{\Lambda} \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2}.$$

and

$$\tilde{K} = K, \quad \text{and} \quad \tilde{u} = u.$$

Applications

- (2) Rescale: $q = \frac{q}{b}$
- (3) Renormalize:

$$(\beta\mathcal{H})'[m'] = V\left(\delta f_b^0 + u\delta f_b^1\right) + \int_0^\Lambda \frac{\mathrm{d}^d\mathbf{q}'}{(2\pi)^d} b^{-d} z^2 \left(\frac{\tilde{t} + Kb^{-2}q'^2}{2}\right) |m'(\mathbf{q}')|^2$$

$$+ uz^4 b^{-3d} \int_0^\Lambda \frac{\mathrm{d}^d\mathbf{q}'_1 \mathrm{d}^d\mathbf{q}'_2 \mathrm{d}^d\mathbf{q}'_3}{(2\pi)^{3d}} \vec{m}'(\mathbf{q}'_1)\cdot\vec{m}'(\mathbf{q}'_2)\vec{m}'(\mathbf{q}'_3)\cdot\vec{m}'(-\mathbf{q}'_1-\mathbf{q}'_2-\mathbf{q}'_3).$$

resulting in the recursion relation:

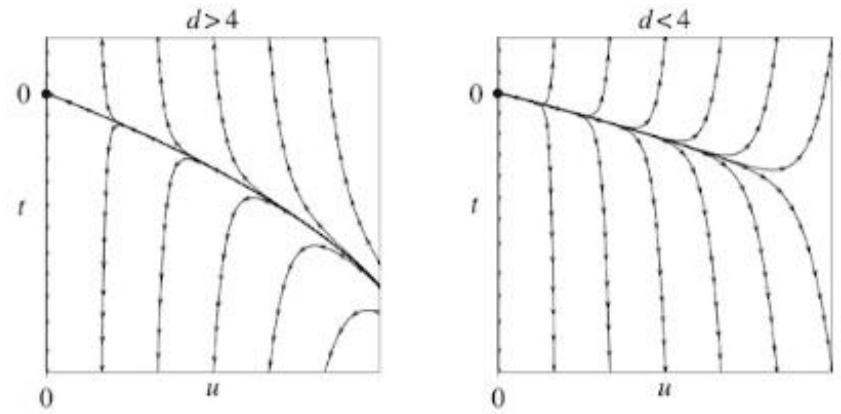
$$\begin{cases} t'_b = b^2\left[t + 4u(n+2)\int_{\Lambda/b}^\Lambda \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\frac{1}{t+Kk^2}\right] \\ u'_b = b^{4-d}u. \end{cases}$$

Applications

For infinetestimal process, we can write:

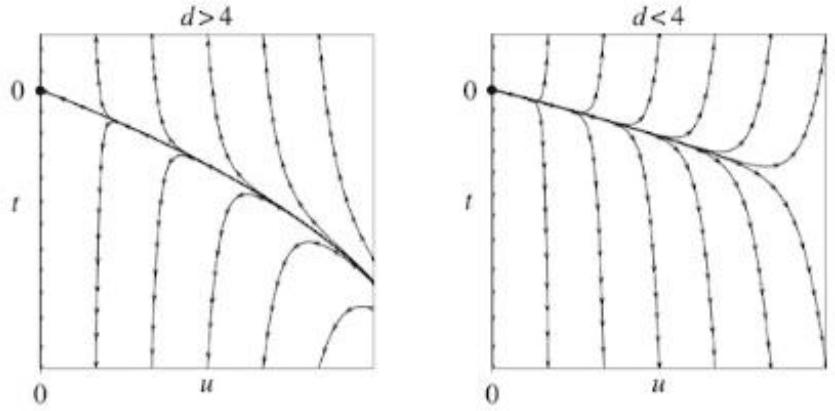
$$\frac{\mathrm{d}}{\mathrm{d}\ell}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}=\begin{pmatrix}2 & \dfrac{4(n+2)K_d\Lambda^{d-2}}{K}\\ 0 & 4-d\end{pmatrix}\begin{pmatrix}\delta t\\ \delta u\end{pmatrix}.$$

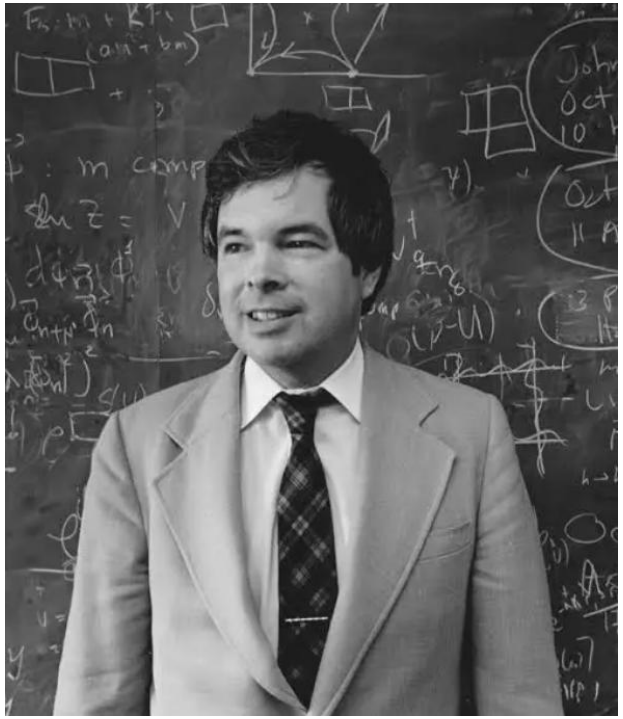
And the RG flow is:



Applications

- For $d > 4$, there is another fixed point other than origin, implying the existence of a critical point. For $d < 4$, however, no such point exists at first order. We must go to *second order* perturbation to see the existence of the critical point. This was first done by K. Wilson and was awarded Nobel Prize in Physics in 1982.





Applications

- 2. Ultraviolet cutoff in QFT:

There is a intimate connection between partition function and Feynman path integral. By setting t in action to $-i\tau$, we can convert Lagrangian to Hamiltonian. In QFT one encounters the divergence of the integral at high-momentum regime. We can use the same RG process to average out short-wavelength modes to avoid divergence.

References

- Statistical Physics of Fields, Mehran Kardar, Cambridge University Press, 2007