GENERALIZED HOOKE'S LAW

1D Hooke's Law

In Mechanics

$$F = -kx$$

In Materials

$$\sigma = E\epsilon$$

stress = stiffness × strain

$$\sigma = \frac{F}{A} \quad \epsilon = \frac{\Delta L}{L}$$

E is Young's modulus

Stress and Strain in 3D

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \qquad \epsilon = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix}$$

σ_{ij} : force in the i-direction on the surface normal to the j-direction.

$$\sigma_{ij} = \sigma_{ji} \quad \epsilon_{kl} = \epsilon_{lk}$$

d.o.f. from 9 to 6

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Generalized Hooke's Law

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}$$

stress component ij depends on strain component kl

$$\sigma_{ij} = \sigma_{ji}$$

$$\epsilon_{kl} = \epsilon_{lk}$$

$$C_{ijkl} = \frac{\partial^2 W}{\partial \epsilon_{ij} \partial \epsilon_{kl}}$$

$$C_{ijkl} = C_{jikl}$$

$$C_{ijkl} = C_{ijlk}$$

$$C_{ijkl} = C_{klij}$$

d.o.f. from 81 to 21

Voigt notation

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{pmatrix}$$

$$\gamma_{ij} = 2\varepsilon_{ij}$$

Cubic Symmetry

$$C_{ijkl} = R_{im}R_{jn}R_{kp}R_{lq}C_{mnpq}$$

90° rotations about the x, y, and z axes leave the crystal unchanged

$$C_{\text{cubic}} = \begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{pmatrix}$$

$$\boxed{\text{3 independent constants: } C_{11},\ C_{12},\ C_{44}}$$

normal stiffness+normal coupling+shear stiffness

Isotropic Material

$$C_{ijkl} = A\delta_{ij}\delta_{kl} + B\delta_{ik}\delta_{jl} + D\delta_{il}\delta_{jk}$$

$$C_{ijkl} = C_{ijlk} \quad \Rightarrow \quad B = D$$

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} \qquad C_{\text{iso}} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}$$

$$\boxed{\text{2 independent constants: } \lambda, \mu}$$

Example: Elastic Wave

For a continuous medium, Newton's second law can be written as:

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j}$$

For isotropic material

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u}$$

This is the elastic wave equation in solids.

Longitudinal wave / P-wave

Assume a plane-wave solution

$$u = u_0 e^{i(k\cdot x - \omega t)}$$

The displacement direction is parallel to the propagation direction

$$u_0 \parallel k \quad \Rightarrow \quad \nabla \cdot u \neq 0$$

After substituting into the wave equation

$$\omega^2 = \frac{\lambda + 2\mu}{\rho} k^2$$

$$\boxed{v_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}}$$

Transverse wave / S-wave

The displacement direction is parallel to the propagation direction

$$u_0 \perp k \quad \Rightarrow \quad \nabla \cdot u = 0$$

After substituting into the wave equation

$$\omega^2 = \frac{\mu}{\rho} k^2$$

$$\boxed{v_s = \sqrt{\frac{\mu}{\rho}}}$$

Transverse wave / S-wave

The displacement direction is parallel to the propagation direction

$$u_0 \perp k \quad \Rightarrow \quad \nabla \cdot u = 0$$

After substituting into the wave equation

$$\omega^2 = \frac{\mu}{\rho} k^2$$

$$\boxed{v_s = \sqrt{\frac{\mu}{\rho}}}$$

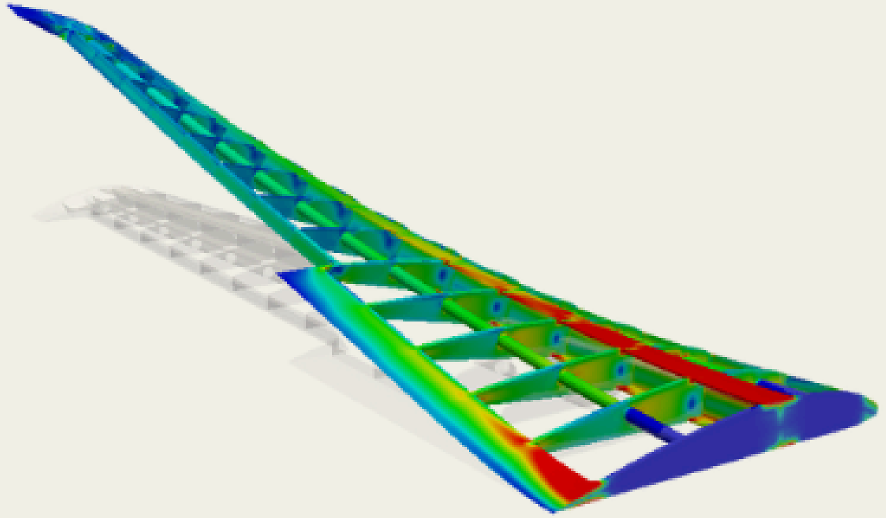
Example: Finite Element Method

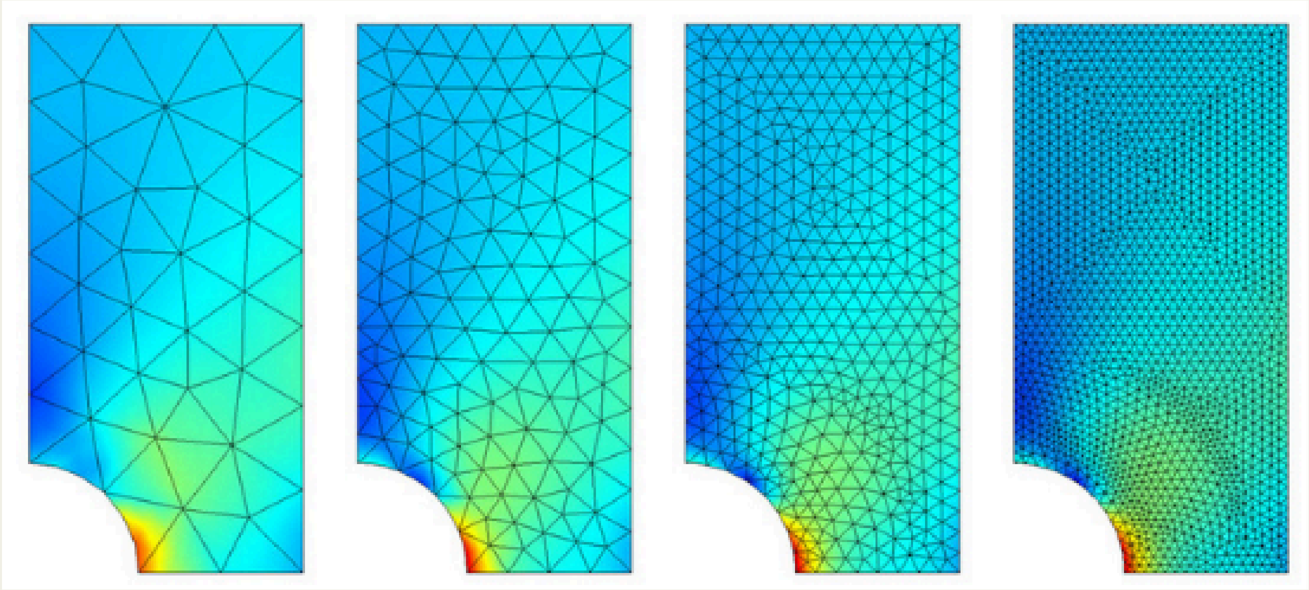
Displacement→Strain→Stress→Force balance

$$\sigma = \mathbf{D}\epsilon$$

$$\mathbf{K}\mathbf{u} = \mathbf{f}$$

$$\mathbf{k}_e = \int_{\Omega_e} \mathbf{B}^T\mathbf{D}\mathbf{B}\, d\Omega$$





THANK YOU