Wiener Process and Stochastic Physics

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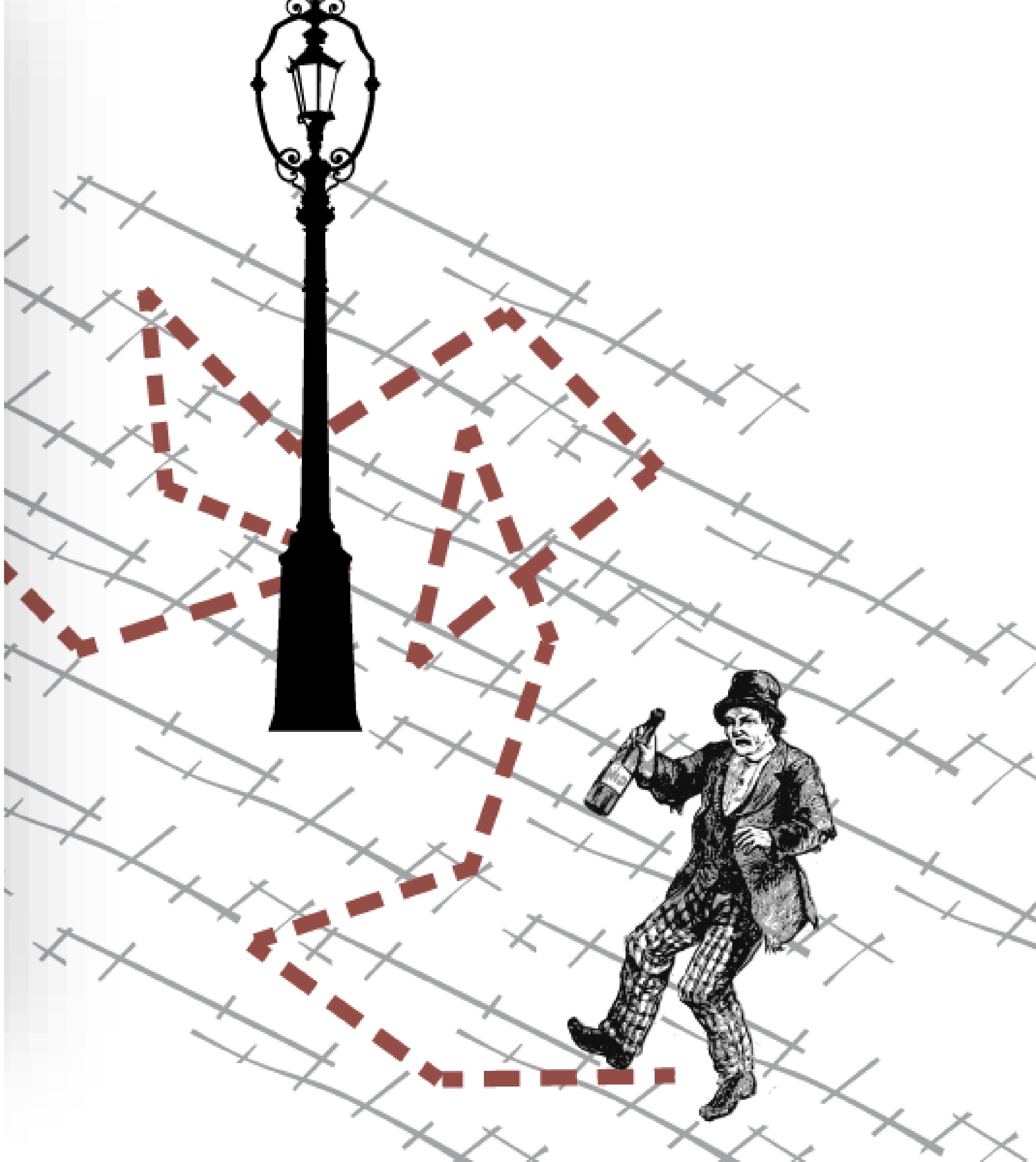
Motivation

Many physical systems exhibit randomness:

- Stock fluctuation
- thermal diffusion
- Particles move randomly in the fluid



1-D Discrete random walk

Definition

$$x_n = \sum_{i=0}^{n} d_i$$

Where

$$d_i = \begin{cases} 1 & prob = 0.5 \\ -1 & prob = 0.5 \end{cases}$$

→

$$\langle x_n \rangle = 0$$

→

$$\langle {x_n}^2 \rangle = n$$

standard deviation is $\sqrt{n}$

Continuous Limit: Wiener Process

- step size $\rightarrow 0$
- time interval $\rightarrow 0$

Random walk converges to: Wiener Process

Wiener process satisfies the following properties:

1. Initial condition W(0)=0
2. Independent Increments
3. The increment over a time interval obeys a normal distribution:

$$W(t) - W(s) \sim N(0, t - s)$$

Which implies that

$$Var[W(t) - W(s)] = t - s$$

fluctuation size $W(t) \sim \sqrt{t}$

Continuous Limit: Wiener Process

4. Continuous Paths

W(t) is continuous in time, but nowhere differentiable

5. If t →∞, By Central Limit Theorem

$$P(x,t)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$$

Please keep the form of the solution in your mind. Let’s see some examples.

Example I
1-D diffusion equation

- Random motion causes particles to spread over time.
- Assume that $P(x,0)=\delta(x)$

$$\frac{\partial P(x,t)}{\partial t} = D\,\frac{\partial^2 P(x,t)}{\partial x^2}$$

The solution is

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}}\, e^{-\frac{x^2}{2Dt}}$$

Then $\langle x^2 \rangle = 2Dt$, Déjà vu?

→The Wiener process provides the stochastic model underlying diffusion.

Example 2: Langevin Equation

- The Langevin equation is Newton's second law with thermal fluctuation

$$m\frac{dv}{dt} = -\gamma v + \eta(t)$$

$\eta(t)$ is random thermal fluctuation satisfied

$$\langle \eta(t) \rangle = 0$$

And

$$\langle \eta(t)\eta(t') \rangle = 2D\delta(t - t')$$

Where

$$\eta(t) = \frac{dW(t)}{dt}$$

Solution

We can rewrite it as:

$$dv = -\gamma v dt + \sigma dW$$

In the stationary state, we have the solution:

$$v(t) = v_0 e^{-\gamma t} + \sigma \int_0^t e^{-\gamma(t-s)} \, dW_s$$

And the variance:

$$\langle v^2 \rangle = \frac{\sigma^2}{2\gamma}$$

Gaussian stationary distribution

$$P(v) = \sqrt{\frac{\gamma}{\pi\sigma^2}} \exp\left(-\frac{\gamma v^2}{\sigma^2}\right)$$

Thanks for listening
Q&A