Ito’s lemma in Stochastic Calculus

An Additional Term in the Derivatives

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Stochastic Differential Equation

$$\frac{dX}{dt} = b(X_t, t) + \sigma(X_t, t) \cdot \textbf{noise}$$

↓

Assume the noise behave like brownian motion

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t$$

$(B_{t+h} - B_t) \sim N(0,h) \quad E[(B_{t+\Delta t} - B_t)^2] = \Delta t$

How should we Integrate this?

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t$$

$$X_t = X_0 + \int_0^t b(X_t, t)dt + \int_0^t \sigma dB_t$$

To see this, we define a function ϕ, where it takes out the ΔB_{t_j}

$$\int \phi(\omega, t) dB(\omega) = \sum e_j(\omega)[B_{t_{j+1}} - B_{t_j}](\omega)$$

For ϕ_1, we set $e_j = B_j$

$$E[\int \phi_1(\omega, t) dB(\omega)] = \sum E[B_{t_j}(\omega)(B_{t_{j+1}} - B_{t_j})] = 0$$

For ϕ_2, we set $e_j = B_{j+1}$

$$E[\int \phi_2(\omega, t) dB(\omega)] = \sum E[B_{t_{j+1}}(B_{t_{j+1}} - B_t)^2]$$

$$= \sum E[(B_{t+1} - B_t)^2] = T$$

Choosing different end points gives us different results

Ito's Integration vs. Stratonovich Integral

Ito's Integration -> $\int f(t)dB_t = \sum f(t_j)(B_{j+1} - B_j)$

Stratonovich Integration -> $\int f(t)dB_t = \sum f(\frac{t_{j+1} + t_j}{2})(B_{j+1} - B_j)$

Ito's Lemma

$$Y_t = g(X_t, t) \qquad dX_t = udt + vdB_t$$

$$dY_t = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dX_t + \frac{1}{2}\frac{\partial g^2}{\partial^2 x}(dX_t)^2$$

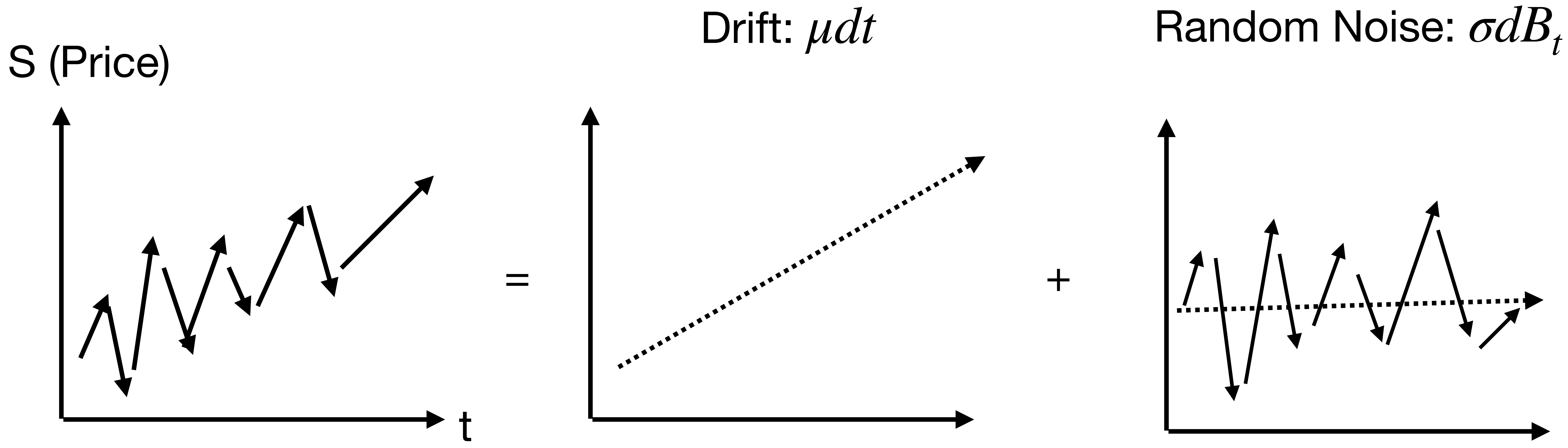
An intuitive reason for this is $E(\Delta B_t^2) = \Delta t$

In fact $(\Delta B_t)^2 = dt$, so

$$dY_t = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dX_t + \frac{1}{2}\frac{\partial g^2}{\partial^2 x}dt$$

Application 1: Modeling a Stock

$$\frac{dS}{S} = \mu dt + \sigma dB_t$$



Price of a Call Option (看漲期權)

Hedge fund would like to eliminate the “risk” of stock using options

Using Ito’s lemma, $$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial^2 C}{\partial S^2}(dS)^2$$

$$(dS)^2 = (\mu S dt + \sigma S dB_t)^2$$

$$dC = (\frac{\partial C}{\partial t} + \mu S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2})dt + \sigma S\frac{\partial C}{\partial S}dB_t$$

Time related term

Stochastic term

Price of a Call Option

$$dC = (\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2})dt + \sigma S \frac{\partial C}{\partial S} dB_t$$

We can create a portfolio that cancel out the randomness by

$$\Pi = -C + S\frac{\partial C}{\partial S}$$

$$d\Pi = -dC + \frac{\partial C}{\partial S}dS = (-\frac{\partial C}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2})dt = r\Pi dt$$

You may then earn the risk free interest without “betting on the randomness”

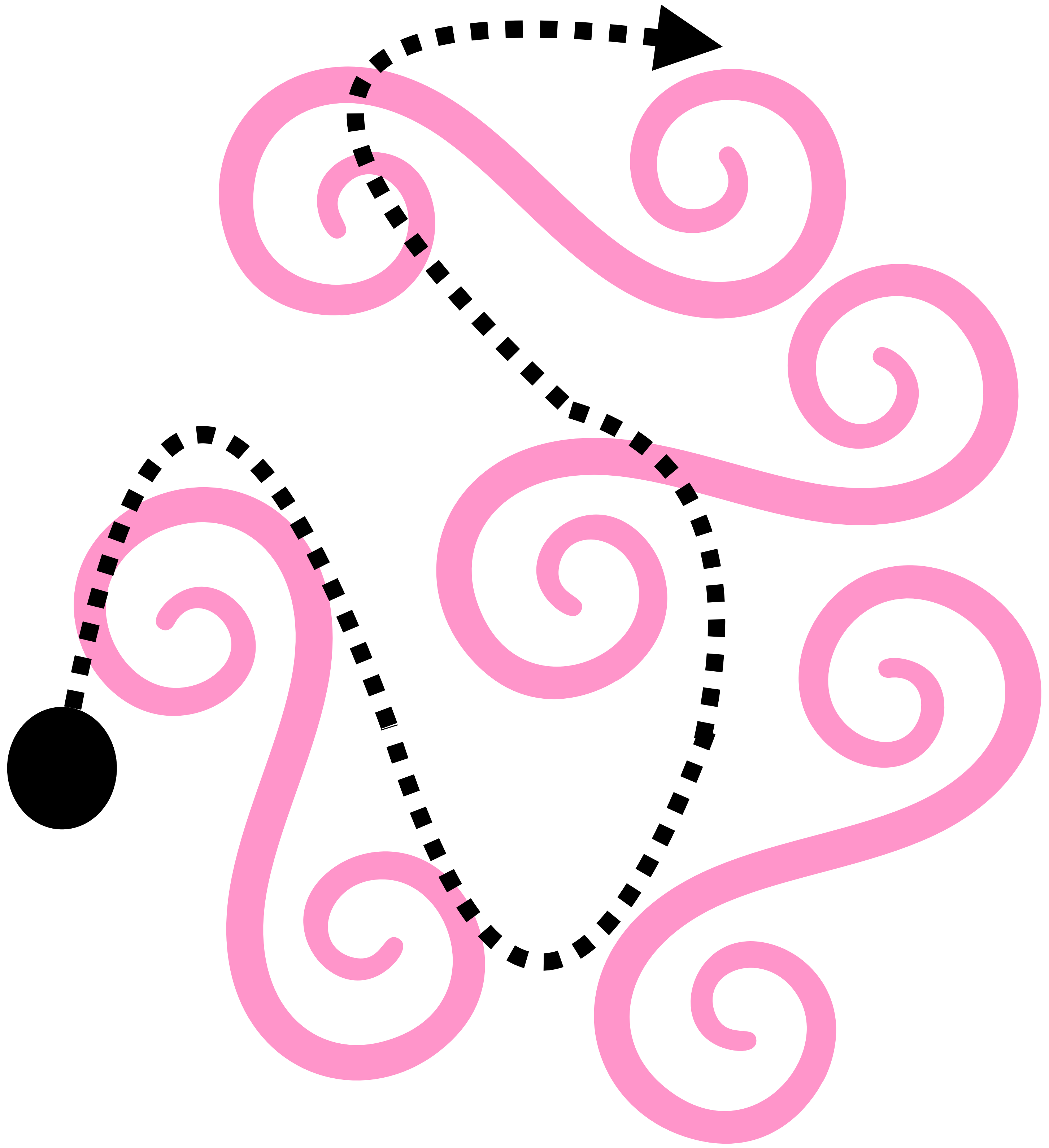
Application 2: Modeling the transport of Turbulence
The Kraichnan Model

$$\langle v^i(t,r)\ v^j(t,r')\rangle = \delta(t-t')D^{ij}(r,r')$$

$$D^{ij}(r) - D^{ij}(0) = \begin{cases} O(r^2) & \text{for } |r| \ll \eta \\ O(r^{\xi}) & \text{for } \eta \ll |r| \ll L \end{cases}$$

$$dR = v(t,R)dt$$

Which is just the diffusion equation :P



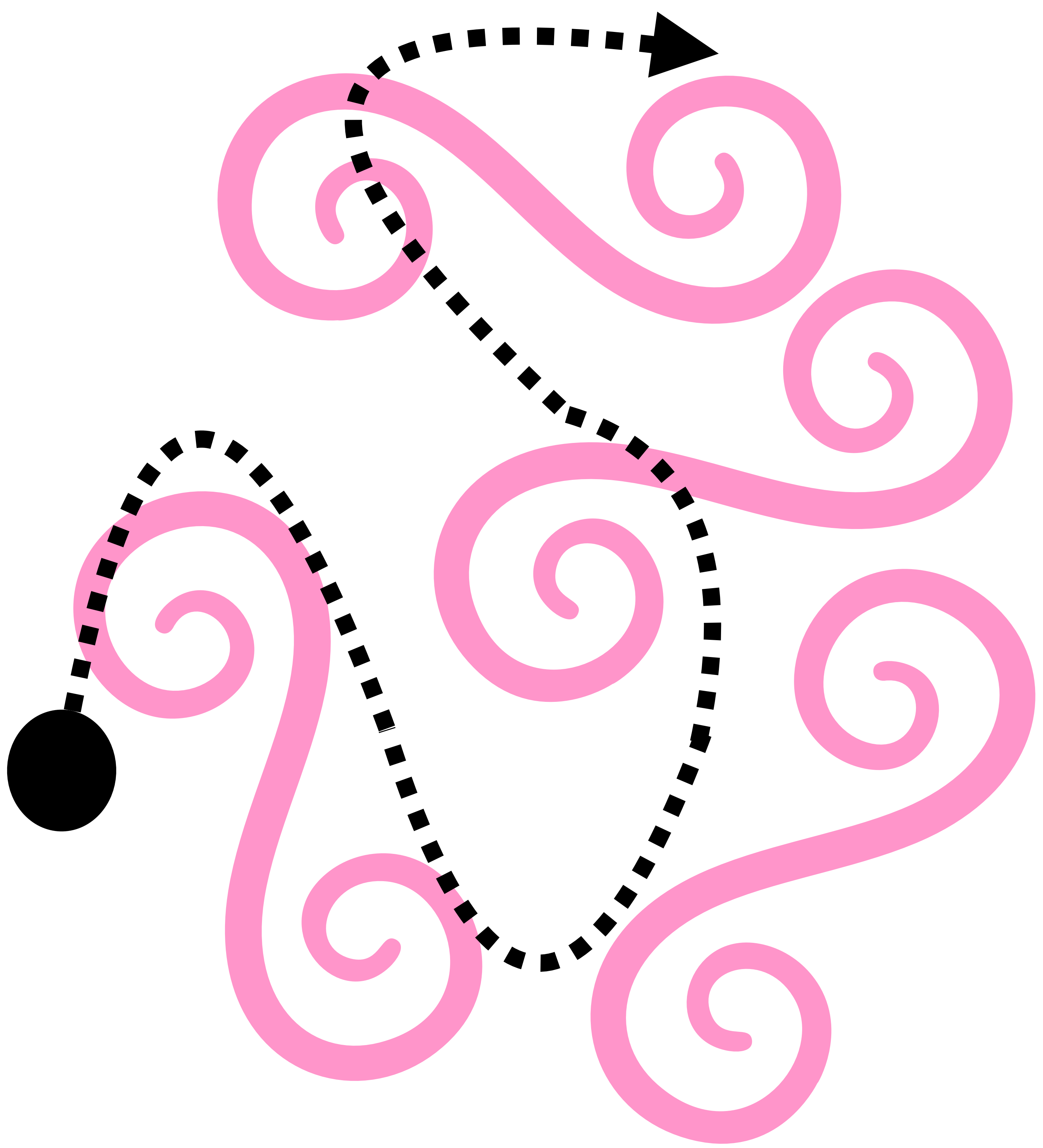
Application 2: Modeling the transport of Turbulence
The Kraichnan Model

$$df(R) = vdt$$

Using Ito's lemma,

$$df(R) = v(R,t)\nabla f dt + \frac{1}{2}D^{ij}\nabla_i \nabla_j f dt$$

$$\frac{d\langle f(R)\rangle}{dt} = \langle \frac{1}{2}D^{ij}\nabla_i \nabla_j f\rangle$$



Which is just the diffusion equation :P

Summary: How do we deal with the Stochastic Differential Equations?

- Taking the expected value -> Kraichnan model
- Eliminating the random part -> Black-Scholes Model