

Ito's lemma in Stochastic Calculus

An Additional Term in the Derivatives

Stochastic Differential Equation

$$\frac{dX}{dt} = b(X_t, t) + \sigma(X_t, t) \cdot \mathbf{noise}$$



**Assume the noise behave like
brownian motion**

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t$$

$$(B_{t+h} - B_t) \sim N(0, h) \quad E[(B_{t+\Delta t} - B_t)^2] = \Delta t$$

How should we Integrate this?

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t$$

$$X_t = X_0 + \int_0^t b(X_t, t)dt + \int_0^t \sigma dB_t$$

How should we Integrate this?

$$\frac{\Delta B_t}{\Delta t} = \frac{B_{t+\Delta t} - B_t}{\Delta t} \sim \frac{N(0, \Delta t)}{\Delta t} = N(0, 1/\Delta t)$$

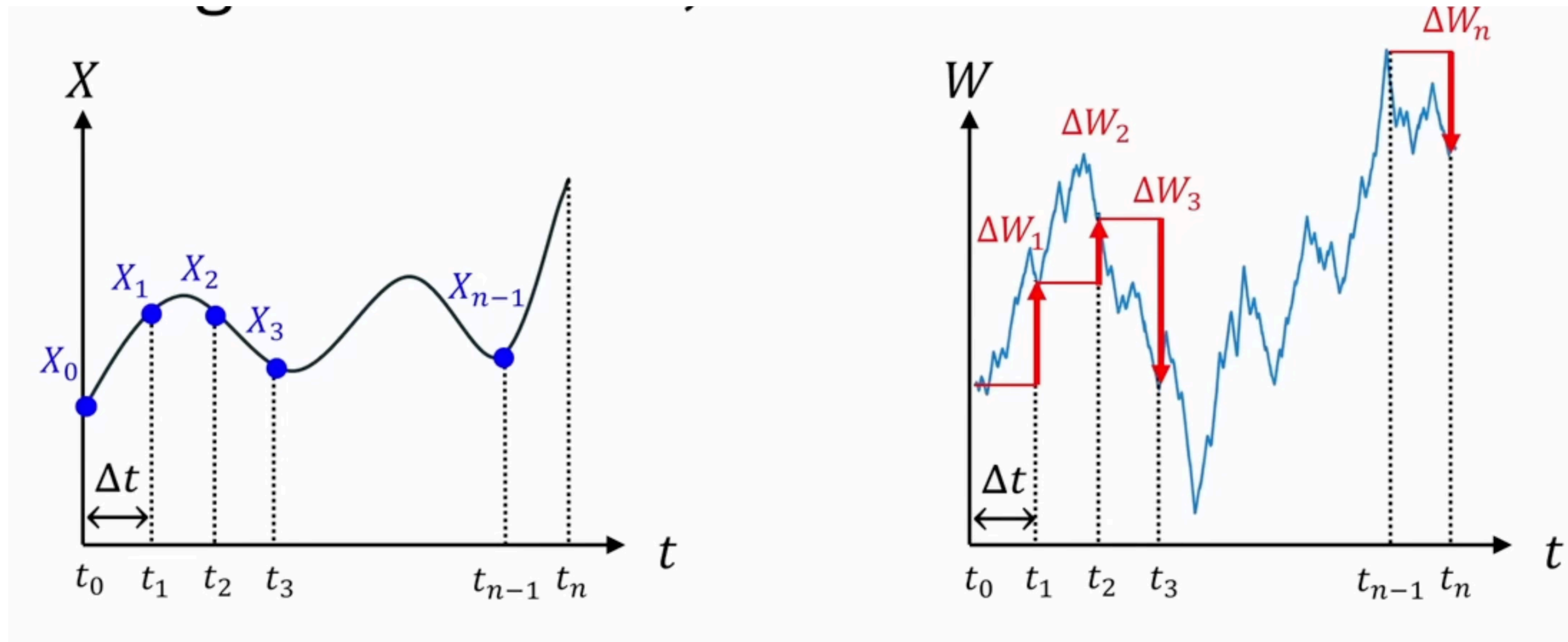
Blows up when $\Delta t \rightarrow 0$, B_t is continuous but non differentiable

We need a new integration law to integrate it!

Ito Integral

$$\int X_s dB_s = \lim_{n \rightarrow \infty} \sum_{i=1}^n X_{t_{i-1}} (B_{t_i} - B_{t_{i-1}})$$

(Left Riemann sum -> Non anticipation)



Ito's Integration vs. Stratonovich Integral

Ito's Integration $\rightarrow \int f(t)dB_t = \sum f(t_j)(B_{j+1} - B_j)$

Stratonovich Integration $\rightarrow \int f(t)dB_t = \sum f\left(\frac{t_{j+1} + t_j}{2}\right)(B_{j+1} - B_j)$

Ito's Lemma

$$Y_t = g(X_t, t) \quad dX_t = udt + vdB_t$$

$$dY_t = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} (dX_t)^2$$

An intuitive reason for this is $E(\Delta B_t^2) = \Delta t$

In fact $(\Delta B_t)^2 = dt$, so

$$dY_t = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dX_t + \frac{1}{2} \frac{\partial^2 g}{\partial x^2} dt$$

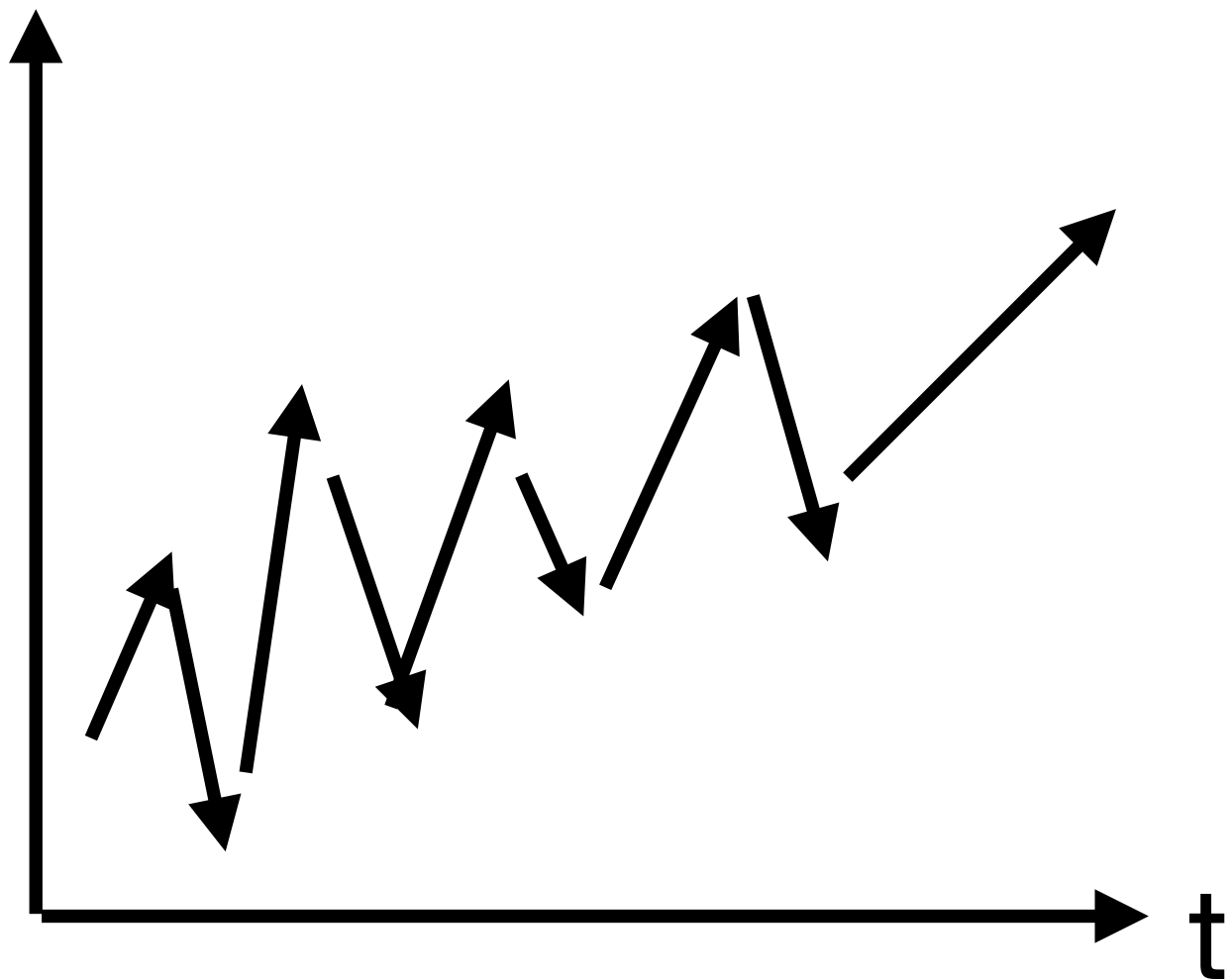
Application 1: Modeling Stock Prices

$$\frac{dS}{S} = \mu dt + \sigma dB_t$$

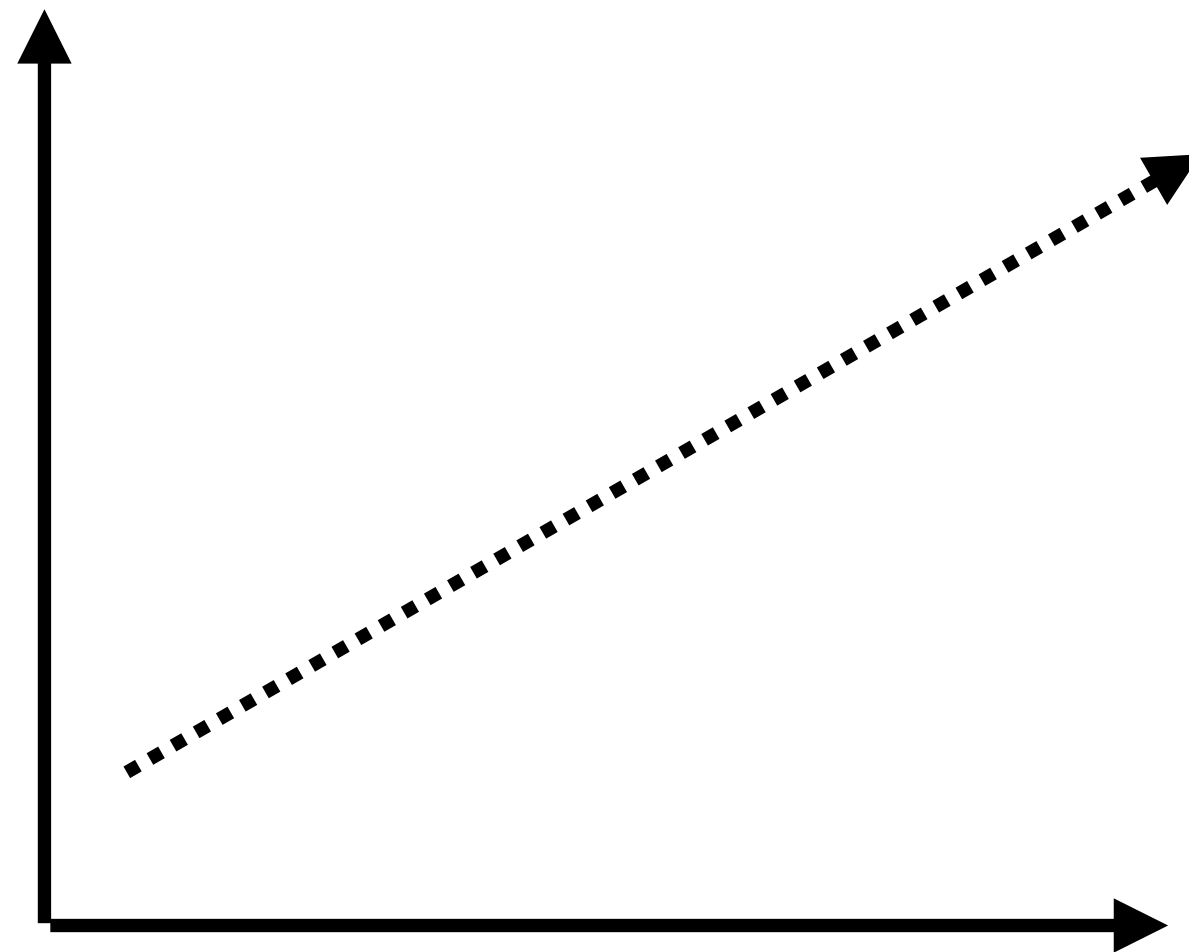
Drift: μdt

Random Noise: σdB_t

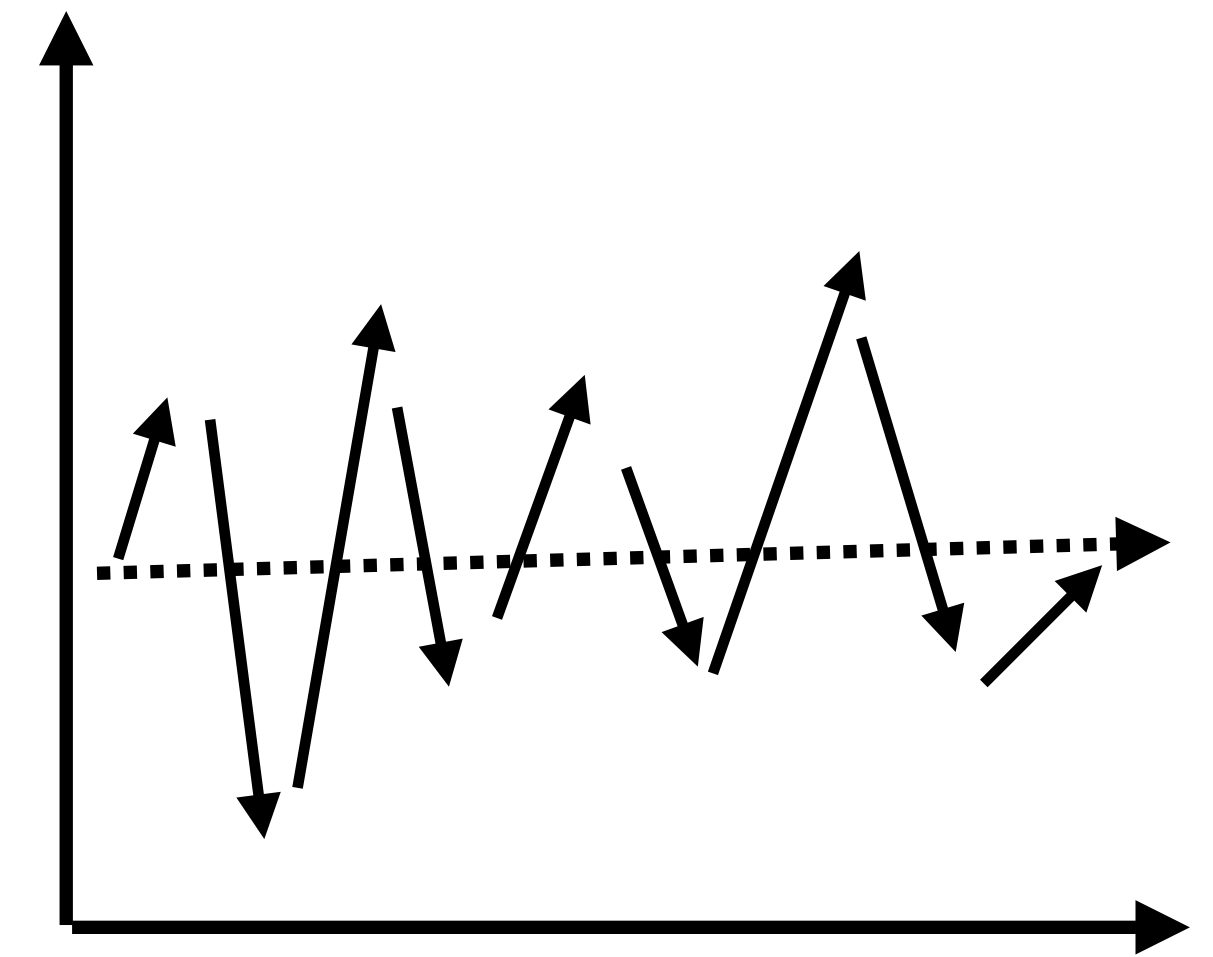
S (Price)



=



+



Price of a Call Option (看漲期權)

Hedge fund would like to eliminate the “risk” of stock using options

Using Ito's lemma,
$$dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS + \frac{\partial^2 C}{\partial S^2} (dS)^2$$

$$(dS)^2 = (\mu S dt + \sigma S dB_t)^2$$

$$dC = \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S} dB_t$$

Time related term

Stochastic term

Price of a Call Option

$$dC = \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \sigma S \frac{\partial C}{\partial S} dB_t$$

We can create a portfolio that cancel out the randomness by

$$\Pi = -C + S \frac{\partial C}{\partial S}$$

$$d\Pi = -dC + \frac{\partial C}{\partial S} dS = \left(-\frac{\partial C}{\partial t} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt = r\Pi dt$$

You may then earn the risk free interest without “betting on the randomness”

Application 2: Modeling the transport of Turbulence

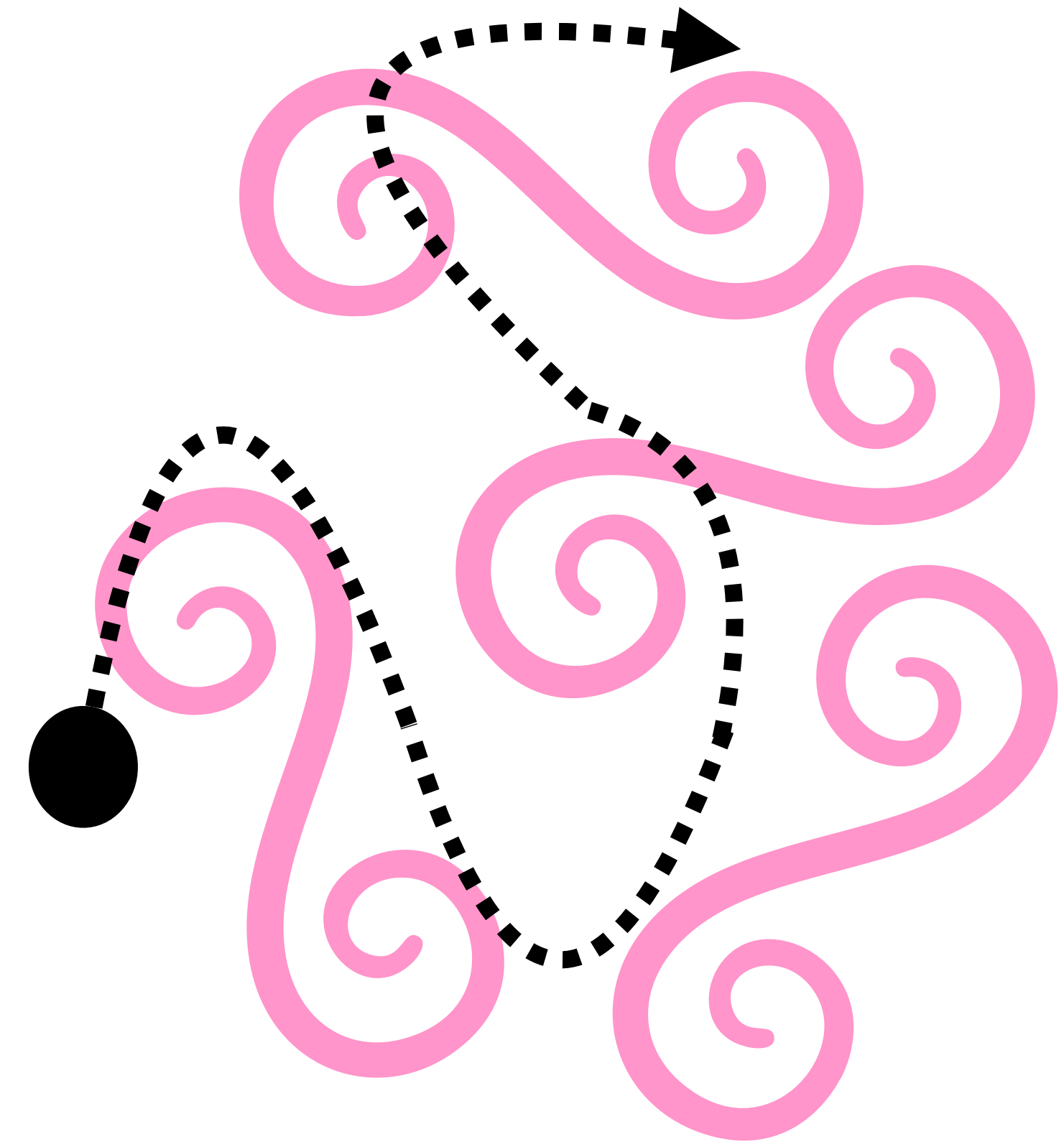
The Kraichnan Model

$$\langle v^i(t, r) v^j(t, r') \rangle = \delta(t - t') D^{ij}(r, r')$$

$$D^{ij}(r) - D^{ij}(0) = \begin{cases} O(r^2) & \text{for } |r| \ll \eta \\ O(r^\xi) & \text{for } \eta \ll |r| \ll L \end{cases}$$

$$dR = v(t, R) dt$$

Which is just the diffusion equation :P



Application 2: Modeling the transport of Turbulence

The Kraichnan Model

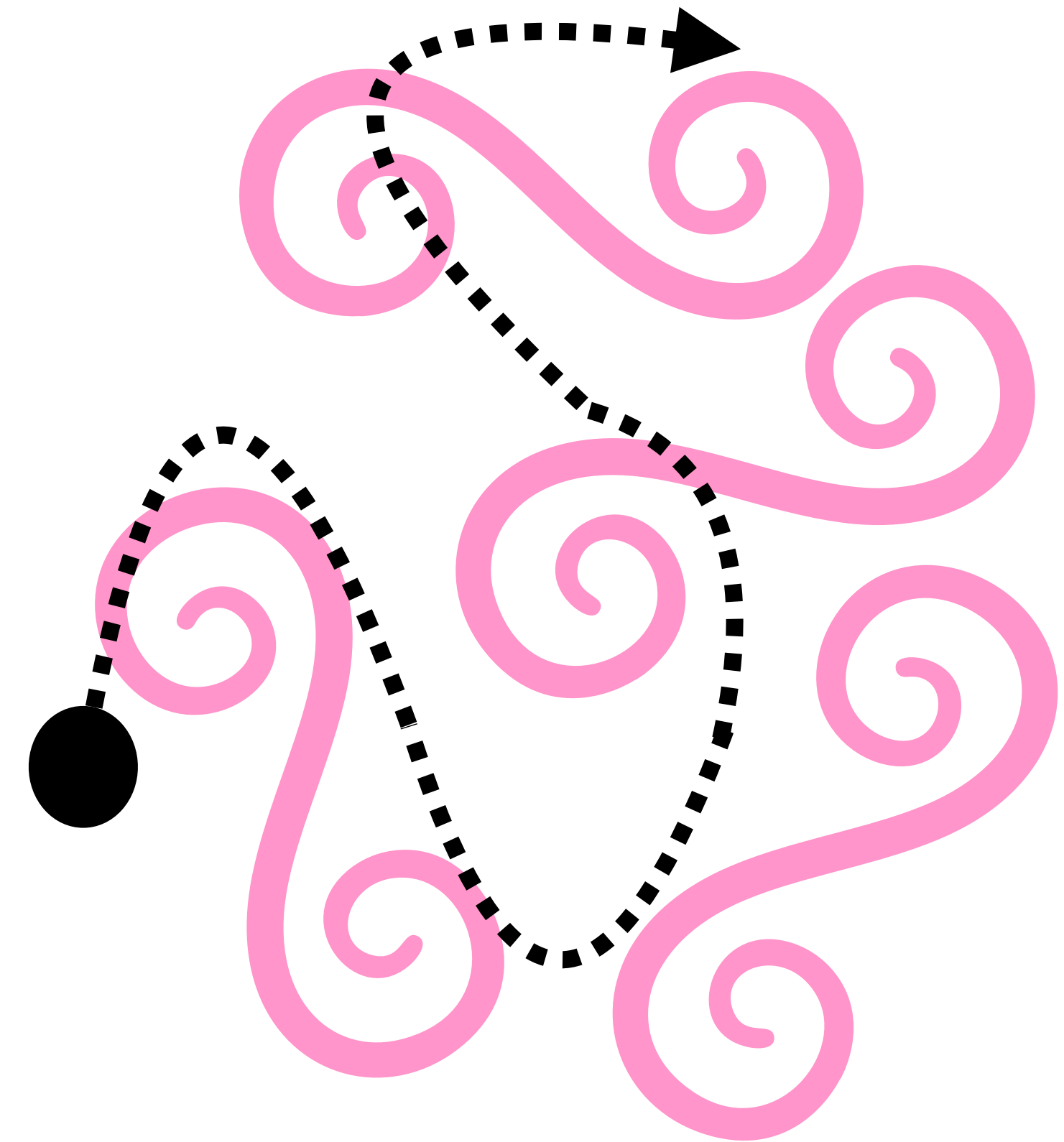
$$df(R) = v dt$$

Using Ito's lemma,

$$df(R) = v(R, t) \nabla f dt + \frac{1}{2} D^{ij} \nabla_i \nabla_j f dt$$

$$\frac{d\langle f(R) \rangle}{dt} = \left\langle \frac{1}{2} D^{ij} \nabla_i \nabla_j f \right\rangle$$

Which is just the diffusion equation :P



Summary: How do we deal with the Stochastic Differential Equations?

- Taking the expected value \rightarrow Kraichnan model
- Eliminating the random part \rightarrow Black-Scholes Model