

a brief exploration into

Ergodic Theory

111022121 蔡翰宗

Recalling Virial Theorem

7.13 Virial Theorem (Optional)

Another important result of a statistical nature is worthy of mention. Consider a collection of particles whose position vectors \mathbf{r}_α and momenta \mathbf{p}_α are both bounded (i.e., remain finite for all values of the time). Define a quantity

$$S \equiv \sum_{\alpha} \mathbf{p}_\alpha \cdot \mathbf{r}_\alpha \quad (7.199)$$

$$\frac{dS}{dt} = \sum_{\alpha} (\mathbf{p}_\alpha \cdot \dot{\mathbf{r}}_\alpha + \dot{\mathbf{p}}_\alpha \cdot \mathbf{r}_\alpha)$$

$$\left\langle \frac{dS}{dt} \right\rangle = \frac{1}{\tau} \int_0^\tau \frac{dS}{dt} dt = \frac{S(\tau) - S(0)}{\tau}$$



$$\langle T \rangle = -\frac{1}{2} \left\langle \sum_{\alpha} \mathbf{F}_\alpha \cdot \mathbf{r}_\alpha \right\rangle$$

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given enough time, the “average” works even in non-periodic system as long as everything is bounded

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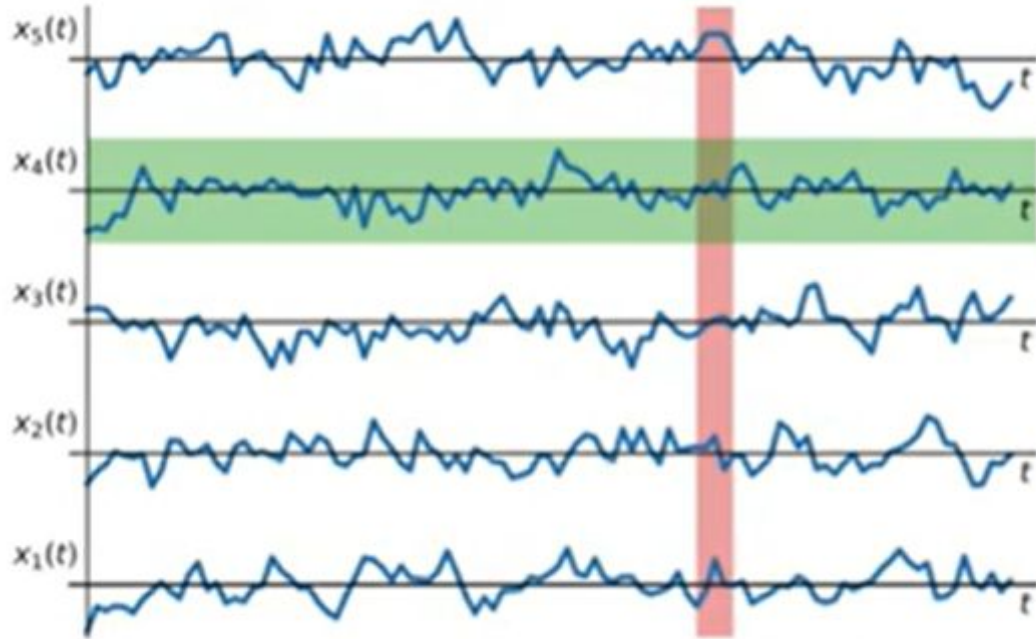


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Average?

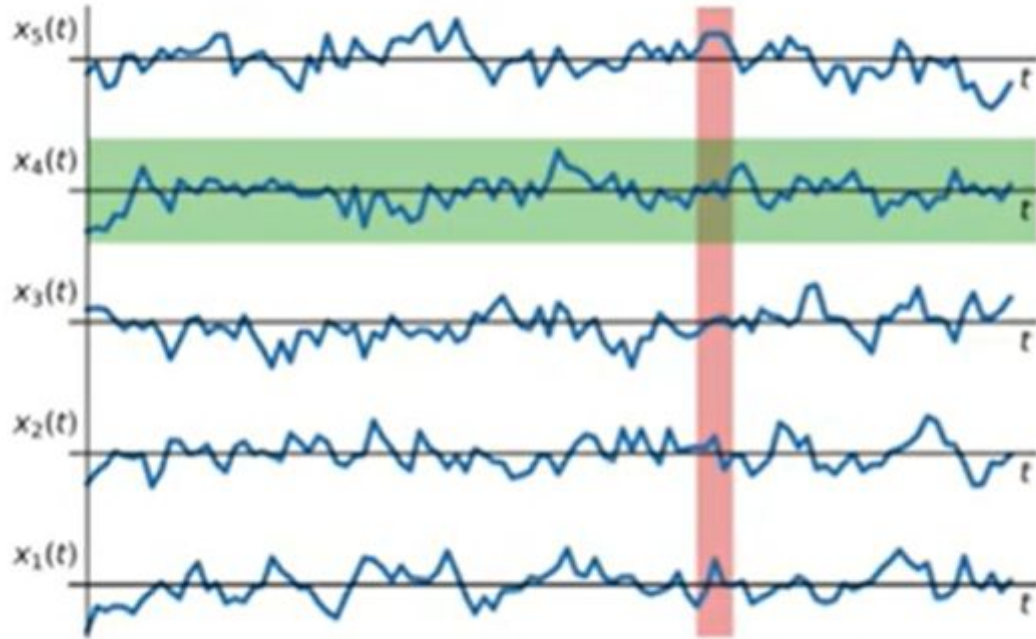


Time average

Phase average

Ensemble average

Average?



Time average



Phase average

Ensemble average

Leading to Birkhoff's ergodic THM

measure space

almost every

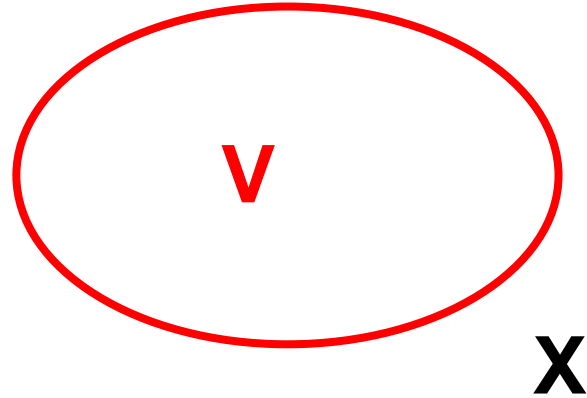
ergodic?

Leading to Birkhoff's ergodic THM

measure space

almost every

ergodic?



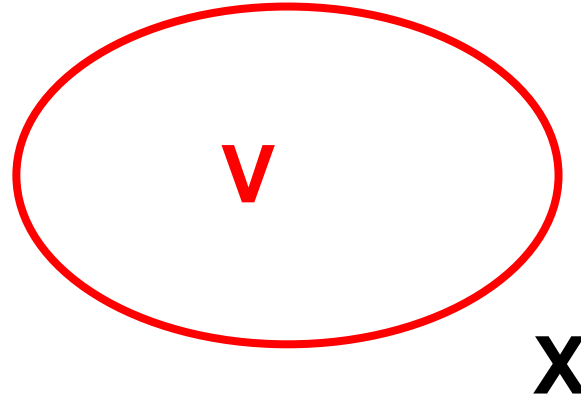
(X, V, μ)

Leading to Birkhoff's ergodic THM

measure space

almost every

ergodic?



for probability measure, $\mu(X) = 1$

Leading to Birkhoff's ergodic THM

measure space

almost every

ergodic?

Let $f: X \rightarrow X$ be μ -measure preserving

$\Rightarrow \forall V \in X$, we got

$\mu(\{x \in V: \exists N \geq 1, \text{ s.t. } \underline{f^n(x) \in V \forall n \in N}\})$

Let $f: X \rightarrow X$ be μ -measure preserving

, $\mu(X) < \infty$

$\Rightarrow f$ is ergodic if $\forall V \in X$, s.t. V is invariant

$\Rightarrow \mu(V) = 0$ or $\mu(V) = \mu(X)$

Leading to Birkhoff's ergodic THM

measure space

almost every

ergodic?

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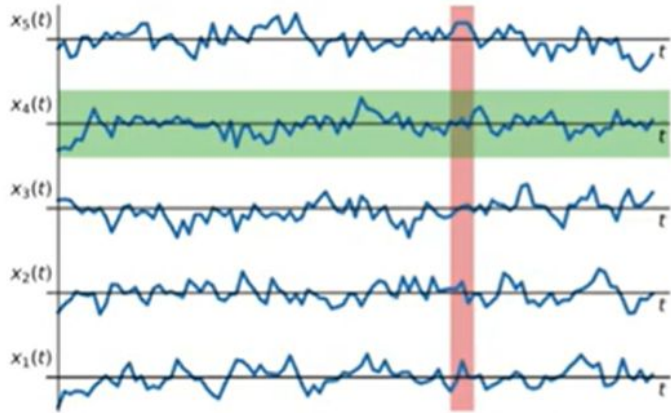
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Birkhoff's ergodic THM



Let $f: X \rightarrow X$, system ergodic w.r.t. μ

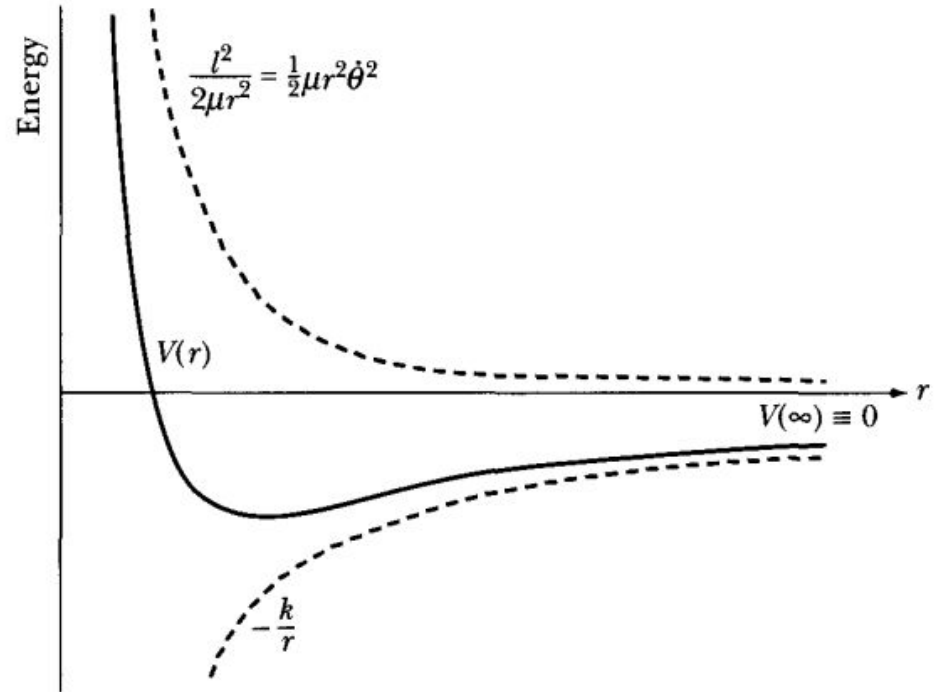
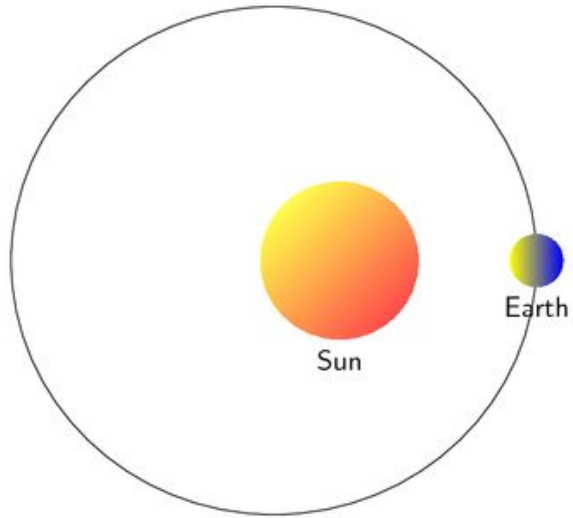
$$\Rightarrow \forall g: X \rightarrow \mathbb{R}, \text{ s.t. } \int_X |g| dx < \infty$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n} \sum g(f^k(x)) = \frac{1}{\mu(X)} \int_X g d\mu$$

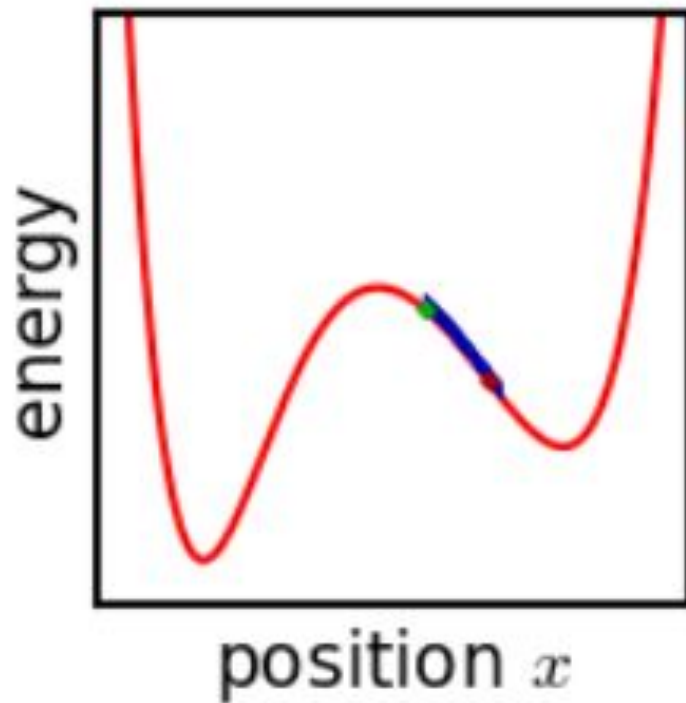
(for almost every $X \in X$)

ergodic? or just diabolic?

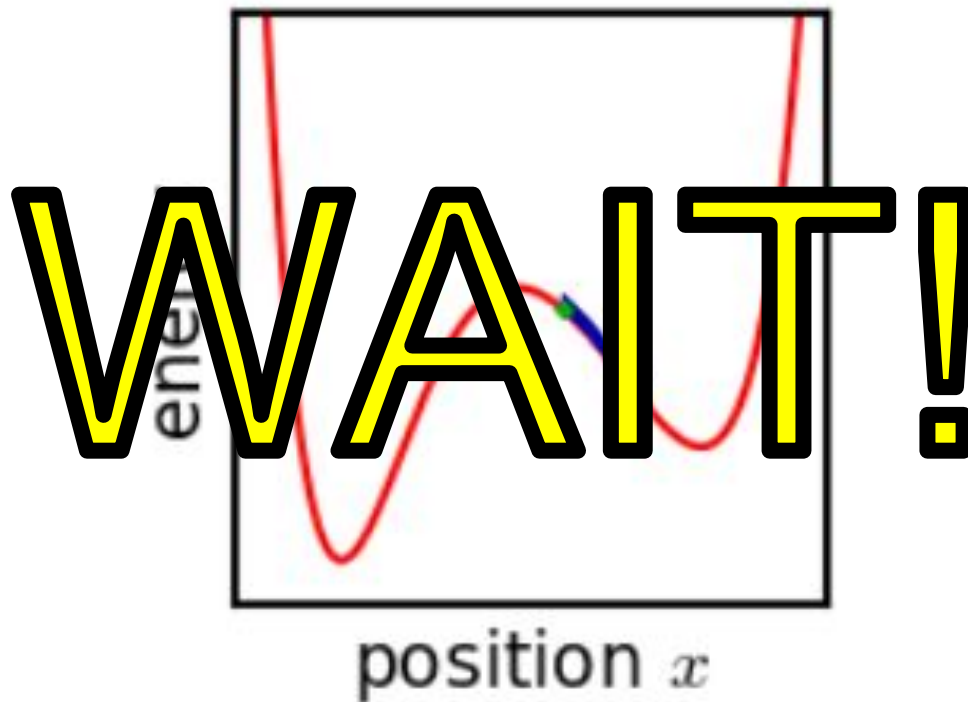
two object



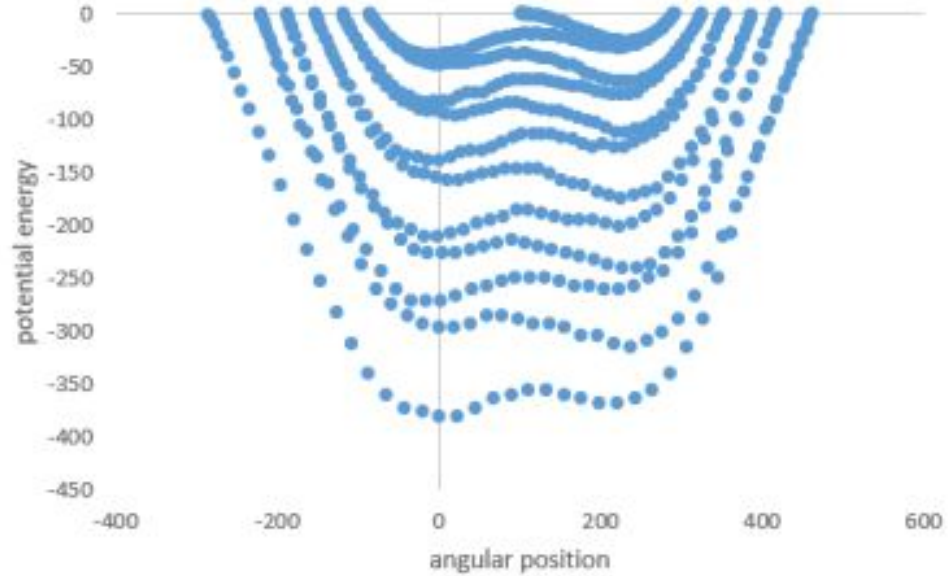
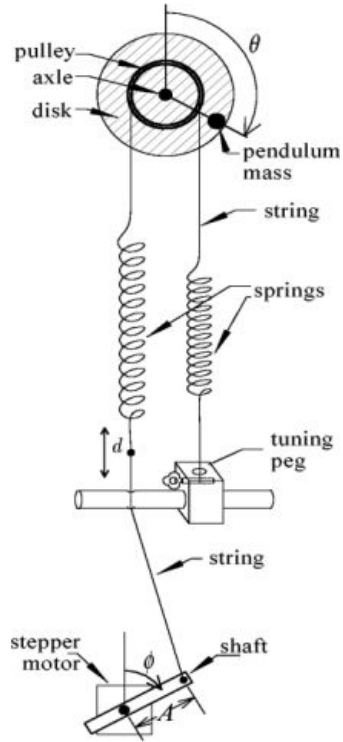
three object



three object

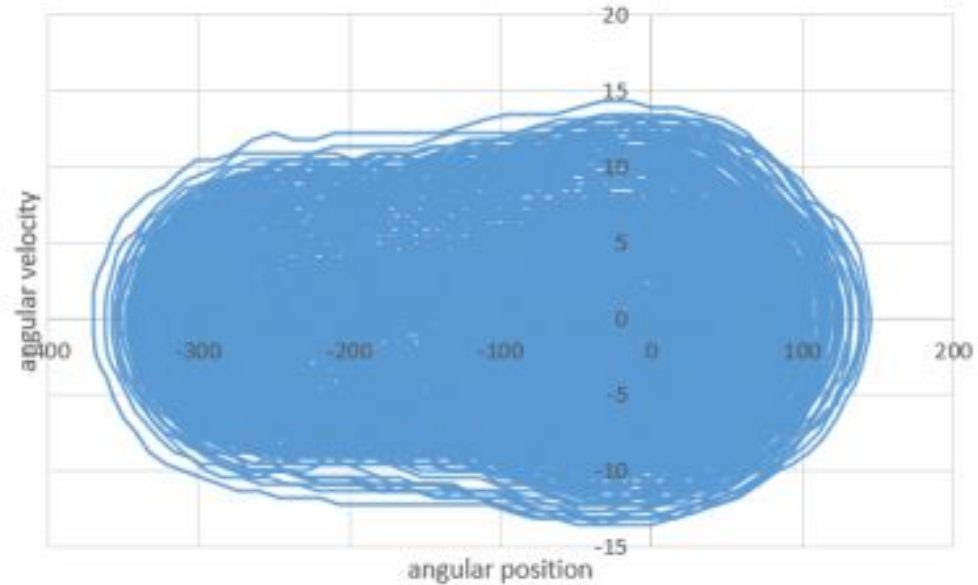
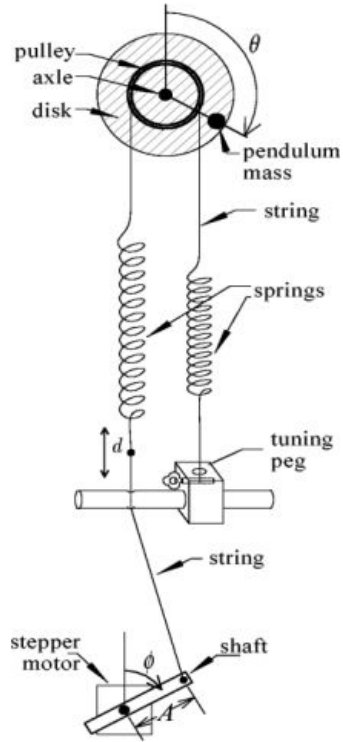


three object \Rightarrow duffing oscillator



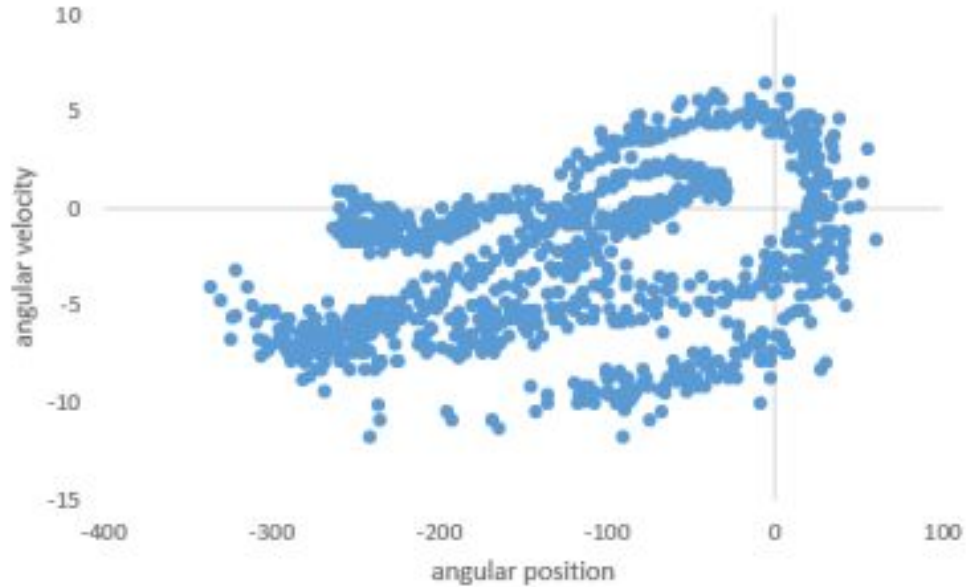
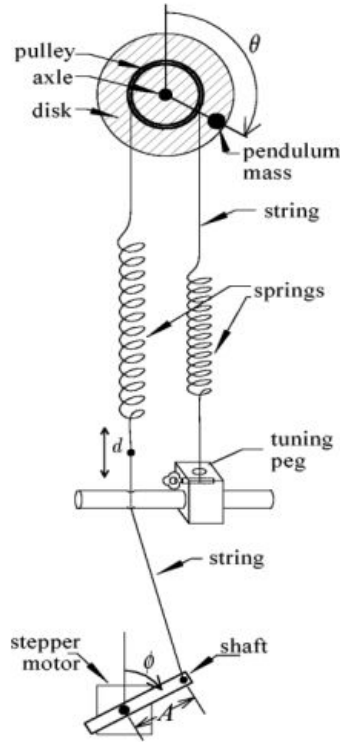
$$\ddot{x} + \delta \dot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega t)$$

three object \Rightarrow duffing oscillator



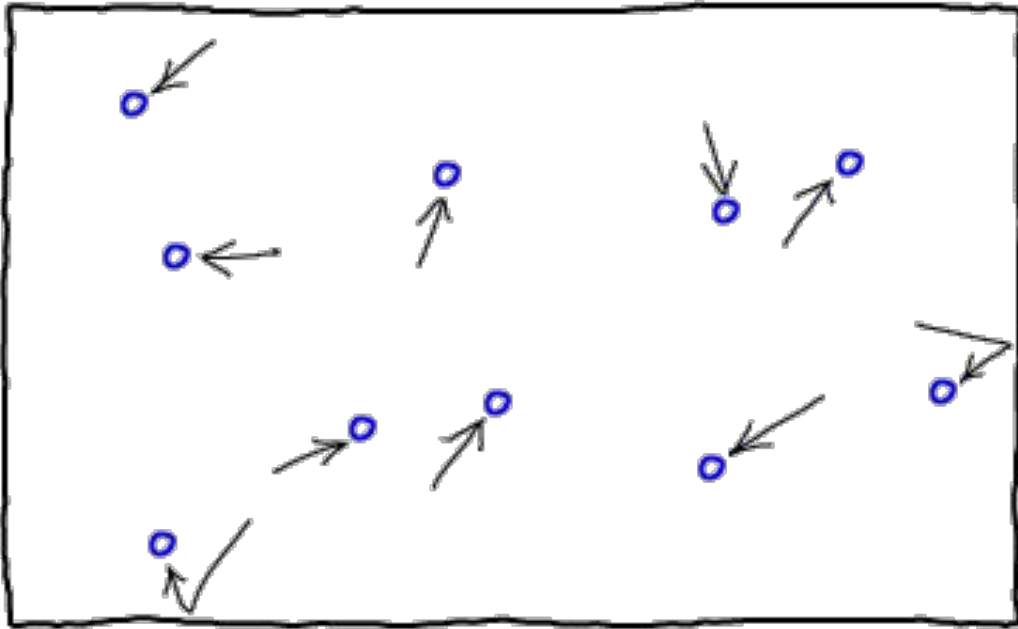
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three object \Rightarrow duffing oscillator



$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma \cos(\omega t)$$

more object



$$\langle T \rangle = \frac{3}{2} NkT_1$$

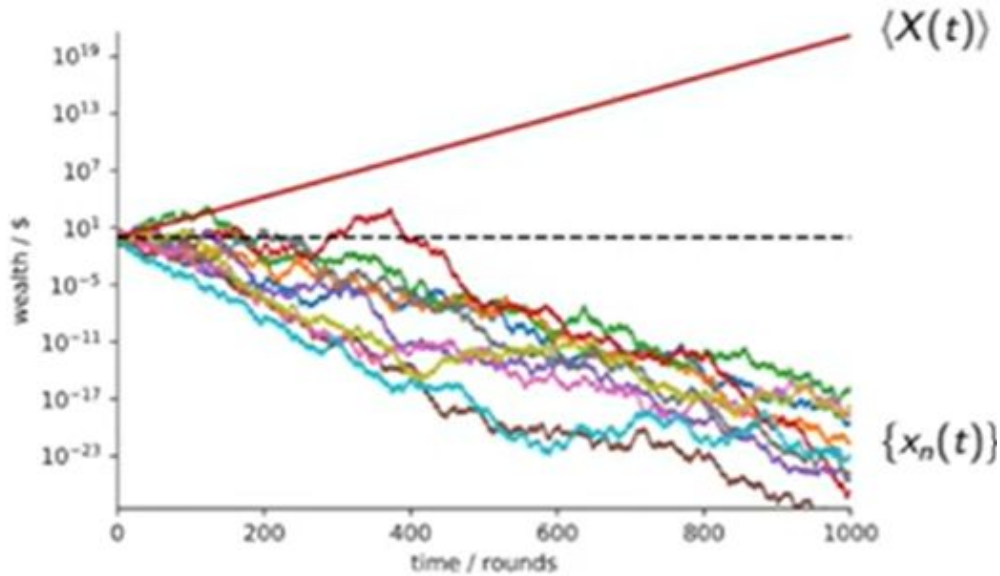
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ergodic? or just chaotic?



ergodic? or just chaotic?

(bonus \Rightarrow non ergodic example)



$$x_n(t+1) = \begin{cases} 1.5x_n(t) & p = 1/2 \\ 0.6x_n(t) & p = 1/2 \end{cases}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$