The muon anomalous magnetic moment as a high-precision probe of the standard model

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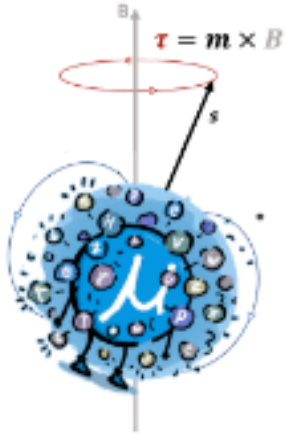

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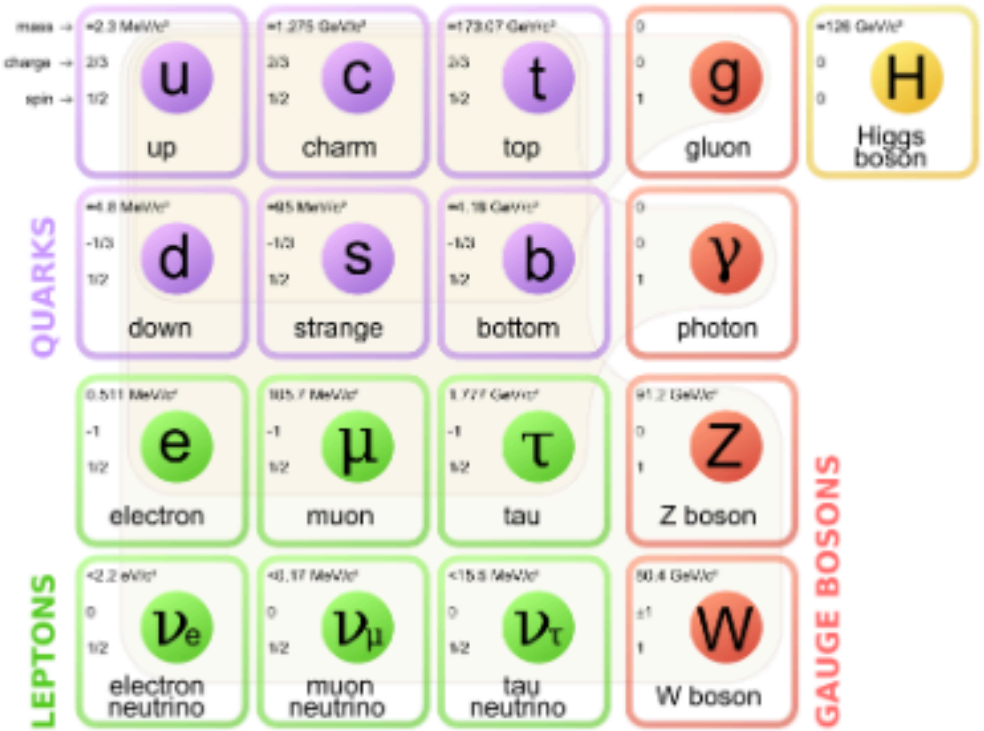
BNL Muon $g-2$ Collaboration, 1999-2006
Fermilab Muon $g-2$ Collaboration, 2021-2025
Aliberti et al., Phys.Rept. 1143 (2025) 1-158 → WP '25
Aoyama et al., Phys.Rept. 887 (2020) 1-166 → WP '20
Boccaletti et al., Nature 653 (2026) → This work or BMW-DMZ '24
Borsanyi et al., Nature 593 (2021) → BMW '20
Borsanyi et al., PRL 121 (2018) 022002 (Editors' Selection) → BMW '17

Introduction and motivation

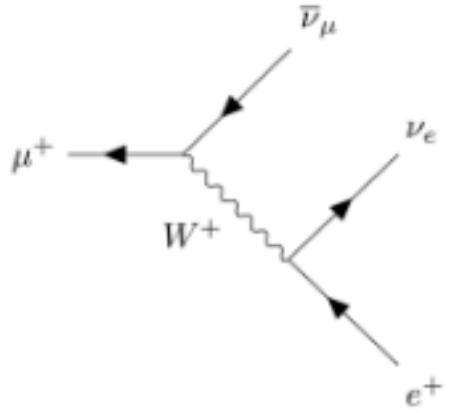
The place of the muon in the Standard Model

The muon (μ) is a cousin of the electron (e):

- both are charged leptons
  - → same spin $= \frac{1}{2}$
  - → same electric charge $= -e$
  - → neither couples directly to the strong interaction
  - → same electromagnetic and weak interactions
- BUT muon couples 207 times more strongly to the Higgs
  - → 207 times more massive
  - → 2 μs lifetime

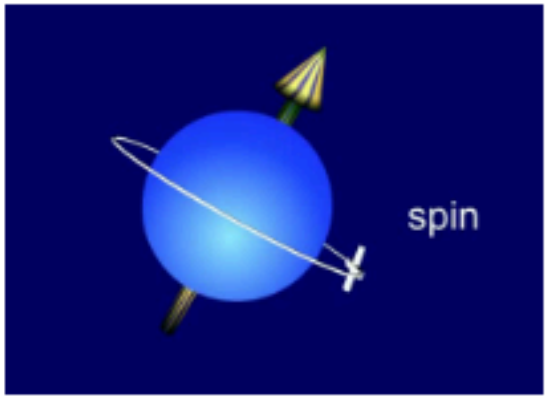


(Wikimedia)



Muons are tiny magnets

A massive elementary particle w/ electric charge and spin behaves like a tiny magnet



(← Silver Swan)

Magnetic moment of the muon

$$\vec{\mu}_\mu = \pm g_\mu \frac{e}{2m_\mu} \vec{S}$$

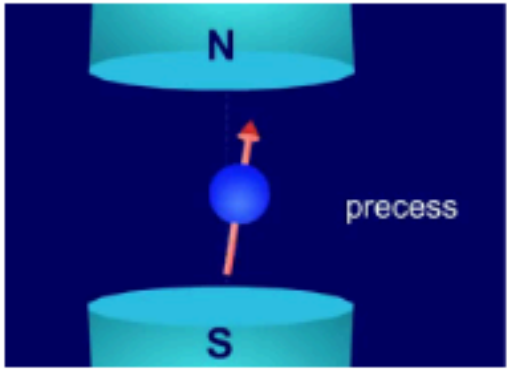
g_μ = Landé factor

In uniform magnetic field $\vec{B}$, $\vec{S}$ precesses w/ angular frequency

$$\omega_S = g_\mu \frac{e}{2m_\mu} |\vec{B}|$$

6×10^9 rotations per second for $|\vec{B}| = 1.45$ T

$\rightarrow$ same principle as for MRI



(Silver Swan)

Crucial point:

- g_μ can be **measured** & **calculated** very, very ... precisely
- measurement = SM prediction ?
  - $\rightarrow$ Yes: another victory for the SM
  - $\rightarrow$ No: we have uncovered new fundamental physics

Interaction with an external EM field: QM

Dirac eq. w/ minimal coupling [1928], w/ ψ, Dirac 4-spinor:

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\vec{\alpha}\cdot\left(c\frac{\hbar}{i}\vec{\nabla} - e_\ell\vec{A}\right) + \beta c^2 m_\ell + e_\ell A_0\right]\psi$$

nonrelativistic limit, i.e. $m_\ell \gg |\vec{p}_\ell|$ ↓

Pauli eq. [1928], w/ ϕ, 2-spinor $\sim$ "large component" of ψ

$$i\hbar\frac{\partial\phi}{\partial t} = \left[\frac{\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e_\ell}{c}\vec{A}\right)^2}{2m_\ell} - \underbrace{\frac{e_\ell\hbar}{2m_\ell}\vec{\sigma}\cdot\vec{B}}_{\vec{\mu}_\ell\cdot\vec{B}} + e_\ell A_0\right]\phi$$

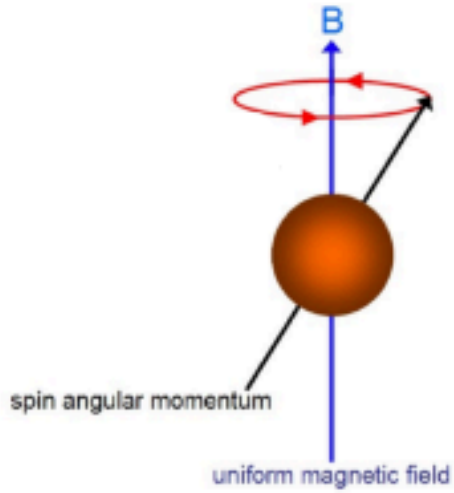
with

$$\vec{\mu}_\ell = g_\ell\left(\frac{e_\ell}{2m_\ell}\right)\vec{S}, \qquad \vec{S} = \hbar\frac{\vec{\sigma}}{2}$$

and

$$g_\ell|_{\text{Dirac}} = 2$$

Exact prediction, no corrections



Early history: the electron

- 1928 : Dirac's new theory predicts the existence of the positron and



$$g_e|_{\text{Dirac}} = 2$$

"That was really an unexpected bonus for me" (P.A.M. Dirac)

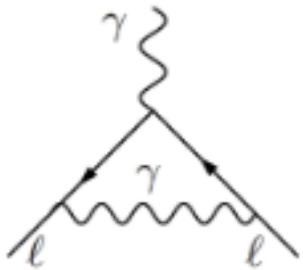


- 1934 : Kinsler & Houston confirm $g_e = 2$, w/ permil precision by studying spectrum of neon atom

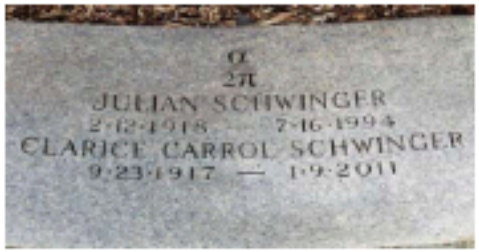
- 1947 : Nafe, Nelson & Rabi, then Kusch & Foley show $g_e > 2$ by 0.1% by measuring hyperfine structure of hydrogen and deuterium

  $\rightarrow$ there is a problem w/ Dirac!

- 1947 : Schwinger understands very quickly that Dirac's theory neglects **quantum fluctuations** and manages to compute them to obtain the **"anomalous"** contribution



$$a_e = \frac{g_e - 2}{2} = \frac{\alpha}{2\pi} = 0.00116\ldots$$



$\rightarrow$ birth of QED and relativistic quantum field theory

Interaction with an external EM field: QFT

Assuming Poincaré invariance and current conservation ($q^\mu J_\mu = 0$ with $q \equiv p' - p$):

$$\langle \ell(p')|J_\mu(0)|\ell(p)\rangle = \bar{u}(p')\left[\gamma_\mu F_1(q^2) + \frac{i}{2m_\ell}\sigma_{\mu\nu}q^\nu F_2(q^2) - \gamma_5\sigma_{\mu\nu}q^\nu F_3(q^2)\right.$$

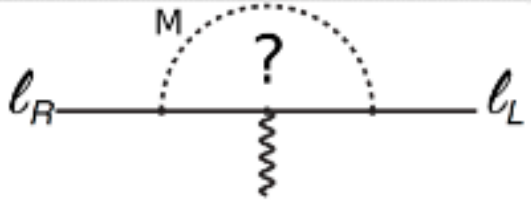
$$\left. +\gamma_5(q^2\gamma_\mu - 2m_\ell q_\mu)F_4(q^2)\right] u(p)$$

- $F_1(q^2)$ → Dirac form factor: $F_1(0) = 1$
- $F_2(q^2)$ → Pauli form factor, magnetic dipole moment: $F_2(0) = a_\ell = \frac{g_\ell - \overbrace{2}^{g_\ell|_{\text{Dirac}}}}{2}$
- $F_3(q^2)$ → $\not{P}$, $\not{T}$, electric dipole moment: $F_3(0) = d_\ell/e_\ell$
- $F_4(q^2)$ → $\not{P}$, anapole moment: $\vec{\sigma}\cdot(\vec{\nabla}\times\vec{B})$

- q^2 dependence of $F_1(q^2)$ and non-zero $F_2(q^2)$ & $F_{3,4}(q^2)$ come from loops
- nevertheless UV finite once lagrangian parameters are renormalized (in a renormalizable theory)

  → "parameter-free" predictions

Why are a_ℓ special?



$$\longrightarrow \qquad \mathcal{L}_{\text{eff}} = -\frac{Qe}{2}\frac{a_\ell}{2m_\ell} F^{\mu\nu}[\bar{\ell}_L \sigma_{\mu\nu} \ell_R] + \text{hc}$$

- a_ℓ, $\ell = e, \mu$, can be computed very accurately in the SM and can also be measured very precisely $\Rightarrow$ excellent tests of SM
- Loop induced $\Rightarrow$ sensitive to possibly new dofs that may be too heavy or too weakly coupled to have been produced directly
- Flavor and CP conserving, chirality flipping $\Rightarrow$ probes mass generating mechanism and complementary to: EDMs, FCNCs (e.g. s and b decays), LHC direct searches, ...
- Chirality flipping $\Rightarrow$ generically, contribution of particle w/ $M \gg m_\ell$

$$a_\ell^{\text{M}} = C\left(\frac{\Delta_{LR}}{m_\ell}\right)\left(\frac{m_\ell}{M}\right)^2$$

- The larger the M, the more precision is needed for revealing the particle's presence
- In EW theory, $M = M_W$, chirality flipping from Yukawa, i.e.

$$\Delta_{LR} = m_\ell \qquad \text{and} \qquad C \sim \frac{\alpha}{4\pi \sin^2\theta_W}$$

- In BSM, can have enhancements: e.g. SUSY $M = M_{\text{SUSY}}$ and $C \sim \alpha/(4\pi \sin^2\theta_W)$ & $\Delta_{LR} = (\mu/M_{\text{SUSY}}) \times \tan\beta \times m_\ell$; or radiative m_ℓ model, $\Delta_{LR} \simeq m_\ell$, $C \sim 1$ and $M = M_{\text{N}\Phi}$

Why is a_μ special?

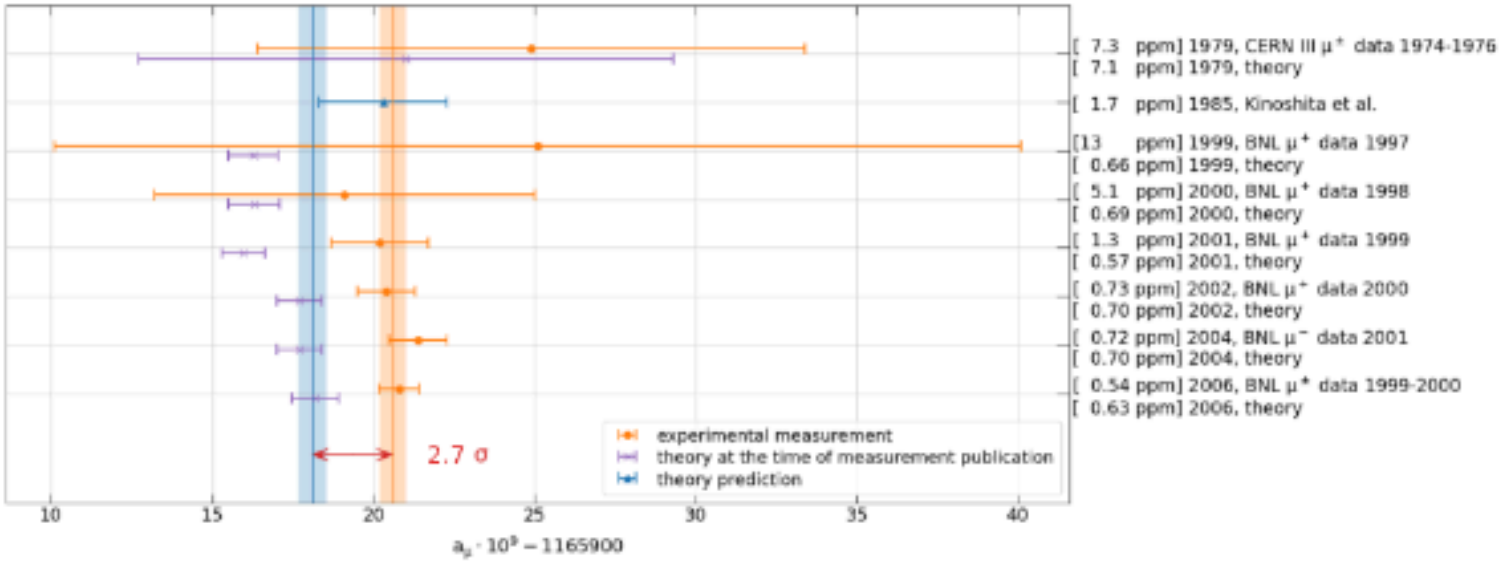
$m_e : m_\mu : m_\tau = 0.0005 : 0.106 : 1.777\,\text{GeV}$ $\quad$ $\tau_e : \tau_\mu : \tau_\tau = \text{“}\infty\text{”} : 2.\cdot 10^{-6} : 3.\cdot 10^{-13}\,\text{s}$

- a_μ is $(m_\mu/m_e)^2 \sim 4.3 \times 10^4$ times more sensitive to new Φ than a_e
- a_τ is even more sensitive to new Φ, but is too shortly lived
- τ_μ small but manageable

A (very) brief history of a_μ

- 1960 : despite $\tau_\mu \sim 2\,\mu$s, Garwin et al manage to measure $g_\mu \simeq 2$
- $>$ 1960 : measurement of a_μ progressed in // with the development of the SM

[Adapted from G. Venanzoni. Colored error bands not relevant here. 2.7σ is between 2006 experiment and theory]



- 2006 : BNL final report [PRD 76, 2006]

$$a_\mu^{\text{exp}} = 11659208.0(6.3) \times 10^{-10}\ [0.54\,\text{ppm}], \qquad a_\mu^{\text{SM}} = 11659180.0(7.3) \times 10^{-10}\ [0.62\,\text{ppm}]$$

$$\Delta a_\mu^{\text{exp-SM}} = 26.1(9.4) \times 10^{-10}$$

$\longrightarrow$ 2.7σ discrepancy was **too small** to claim new physics, but **too large** to ignore ($\sim 2\times$ weak contribution!)

Muon: recent history

To decide on possible presence of new fundamental physics:

Improve the measurement



Move BNL magnet to Fermilab & significantly ugprade experiment

⇒ April 7, 2021, first results (run 1): 6% of planned data and 20% more precise than BNL [Abi et al, PRL 126 (2021)]

⇒ August 10, 2023, runs 2/3: runs 1-3, $\sim 6\times$ BNL stats, 0.20 ppm [Aguillard et al, PRL 131 (2023)]

⇒ June 3, 2025, runs 4/5/6: runs 1-6, $\sim 22\times$ BNL stats, 0.127 ppm [Aguillard et al, 2506.03069]

Improve the SM prediction



Important theoretical/experimental effort to improve SM prediction to comparable level of precision

⇒ White Papers from the muon $g-2$ Theory Initiative w/ reference SM predictions [WP '20, WP '25]

⇒ New measurements of $\sigma(e^+e^- \to \text{hadrons})$ to improve determination of QCD contribution that limits SM prediction precision

⇒ New dispersive approaches for important but much smaller QCD contribution

⇒ *Ab-initio* supercomputer calculations of all highly nonlinear QCD contributions

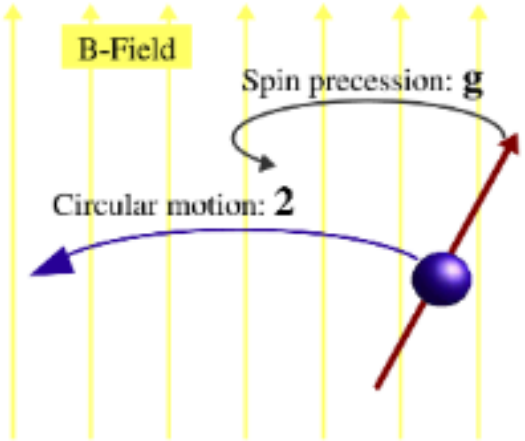
$$a_\mu^{\text{exp}} = a_\mu^{\text{SM}} \; ?$$

Experimental measurement of a_μ

Muon g-2 collaborations at CERN, Brookhaven National Laboratory, and Fermilab were awarded the 2026 Breakthrough Prize in Fundamental Physics, on April 20

For their “multi-decade, groundbreaking contributions to the measurement of the muon’s anomalous magnetic moment, pushing the boundaries of experimental precision and igniting a new era in the quest for physics beyond the Standard Model."

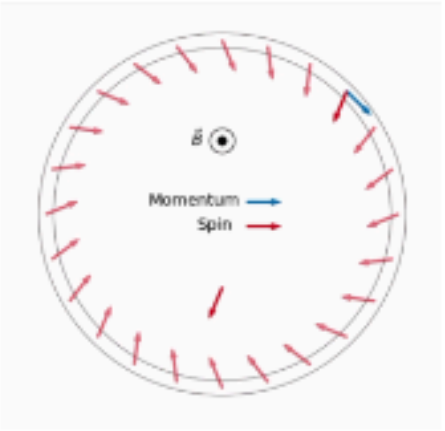
Measurement principle for a_μ



Precession determined by

$$\vec{\mu}_\mu = 2(1 + a_\mu)\frac{Qe}{2m_\mu}\vec{S}$$

$$\vec{d}_\mu = \eta_\mu \frac{Qe}{2m_\mu}\vec{S}$$



$$\vec{\omega}_{a\eta} = \vec{\omega}_a + \vec{\omega}_\eta \simeq -\frac{Qe}{m_\mu}\left[a_\mu \vec{B} - a_\mu\left(\frac{\gamma}{\gamma+1}\right)(\vec{\beta}\cdot\vec{B})\vec{\beta} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\vec{\beta}\times\vec{E} + \frac{\eta_\mu}{2}\left(\vec{E} + \vec{\beta}\times\vec{B}\right)\right]$$

- Experiment measures very precisely $\vec{B}$ with $|\vec{B}| \gg |\vec{E}|$ &

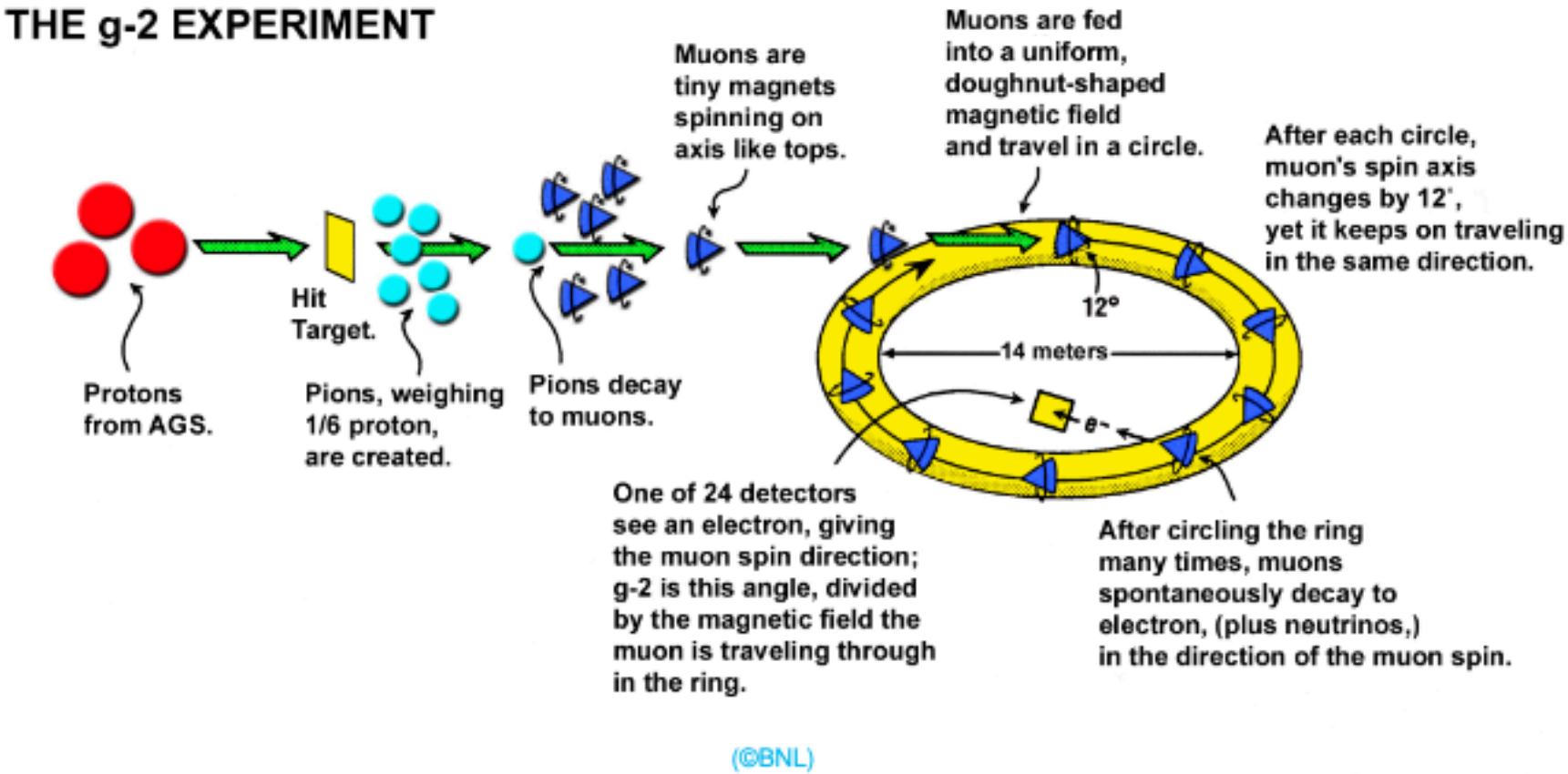
$$\Delta\omega \equiv \omega_S - \omega_C \simeq \sqrt{\omega_a^2 + \omega_\eta^2} \simeq \omega_a$$

since $d_\mu = 0.1(9) \times 10^{-19} e \cdot \text{cm}$ [Bennett et al '09]

- $\vec{\beta}\cdot\vec{B} \simeq 0$, and consider either magic $\gamma = 29.3$ (CERN/BNL/Fermilab) or $\vec{E} = 0$ (J-PARC)

$$\rightarrow \Delta\omega \simeq a_\mu |\vec{B}| \frac{e}{m_\mu}$$

Cartoon of the Fermilab E989 experiment



(©BNL)

However, for experiment to work, need to effectively:

1. produce a highly polarized muon beam
2. detect the direction of the decaying muon's

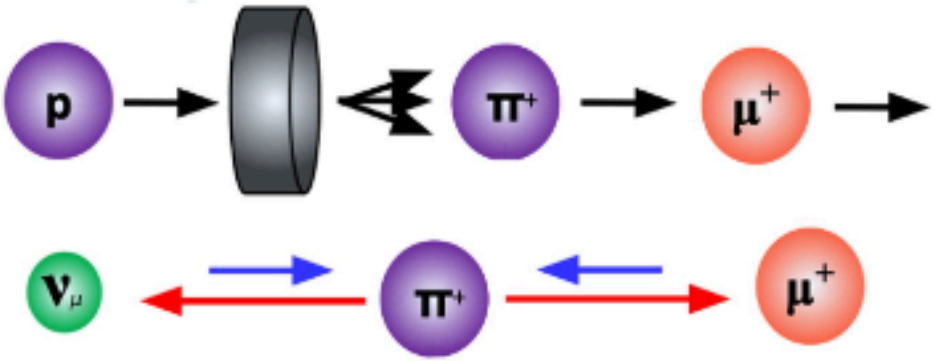
→ enters the parity violation (PV) of the weak interaction

The two miracles of parity violation

Parity violation of weak interaction, $m_{\nu_{e,\mu}} \simeq 0$, momentum & angular-momentum conservation

$\rightarrow$ 2 miracles [Lee & Yang '57]

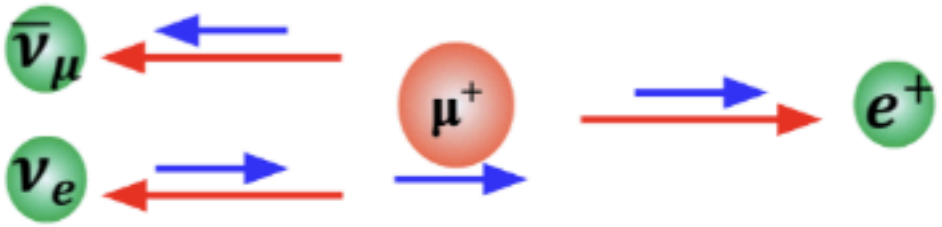
Miracle 1: μ are produced highly polarized



- $\pi^+ \rightarrow \mu^+ \nu_\mu$ w/ $\sim$ 100% branching ratio
- **rest frame:** μ^+ produced with negative helicity
- **lab frame:** fastest μ^+ have negative helicity
- selecting those μ^+ gives 97% negatively polarized beam

The two miracles of parity violation

Miracle 2: μ decays are "self-analyzing"



- $\mu^+ \to e^+ \nu_e \bar{\nu}_\mu$ w/ $\sim 100\%$ branching ratio
- **rest frame:** e^+ produced is most energetic when its momentum is opposite those of ν_e & $\bar{\nu}\mu$
- **lab frame:** no helicity suppression if p_e^{lab} in direction of μ^+ spin
  $\to$ more high-E e^+ produced in μ^+ spin direction
- so, count # of e^+ w/ $E_{e^+} \geq E_{\text{cut}}$, $N_{e^+}(t)$, as function of time t
- then, as μ^+ spin precesses ...
- ... $N_{e^+}(t)$ oscillates from maximum (μ^+ spin aligned with p_μ) to minimum (anti-aligned)
- ... w/ wanted anomalous precession frequency !!!

The two miracles of parity violation

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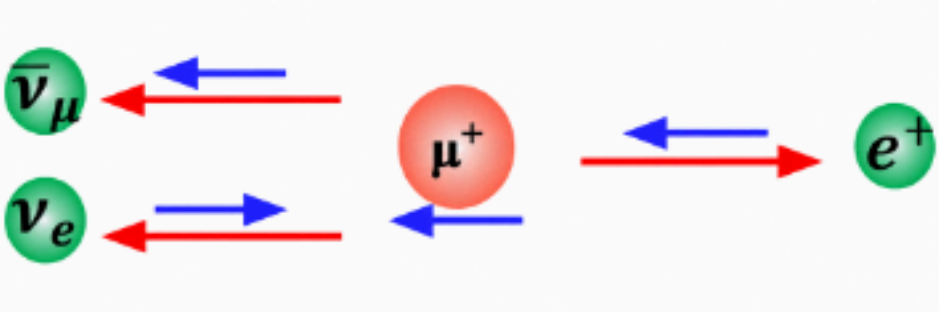

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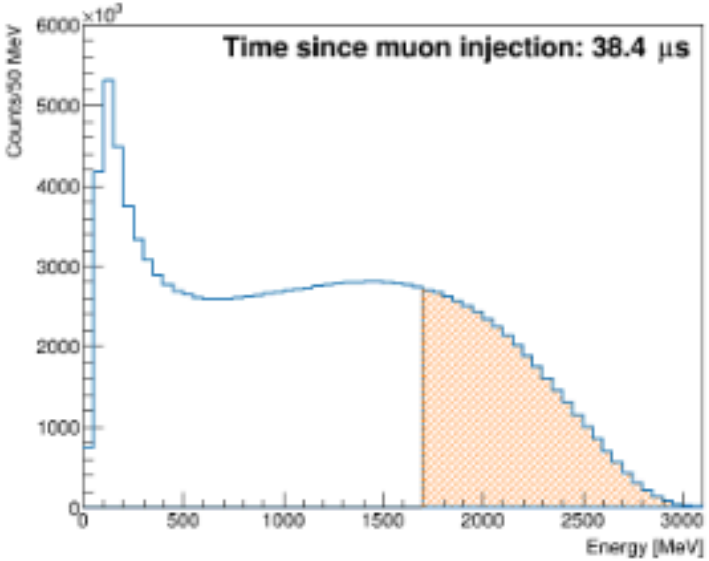
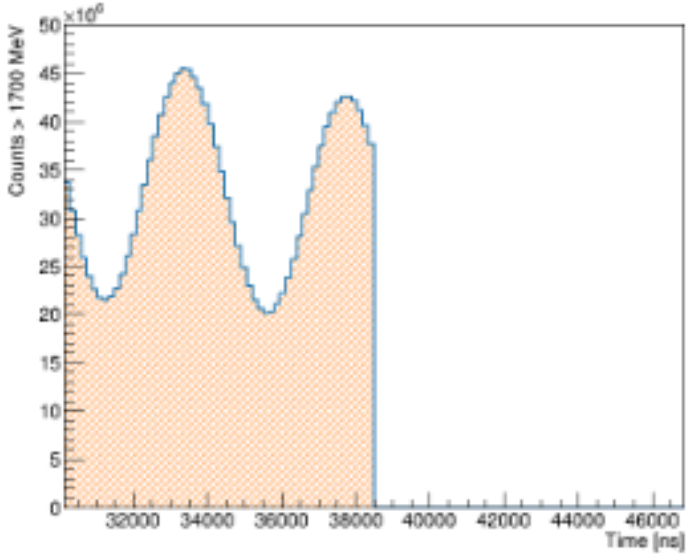
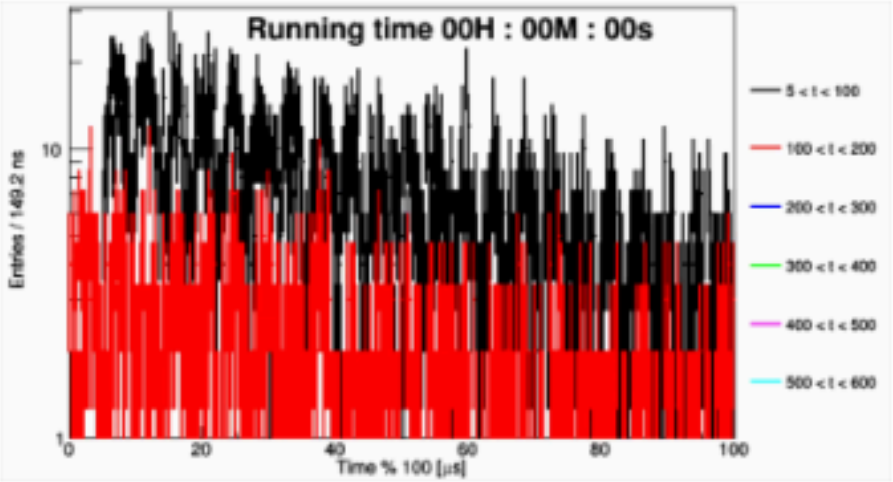
Fermilab E989 ω_a^m measurement

Real data from Run-3







Fit to 5-parameter function

$$N_{e^+}(t) = N_0 e^{-t/\gamma\tau_\mu}\left[1 - A\cos(\omega_a^m t - \phi)\right]$$

roughly captures exponential decay and $(g-2)$ oscillations, but ...

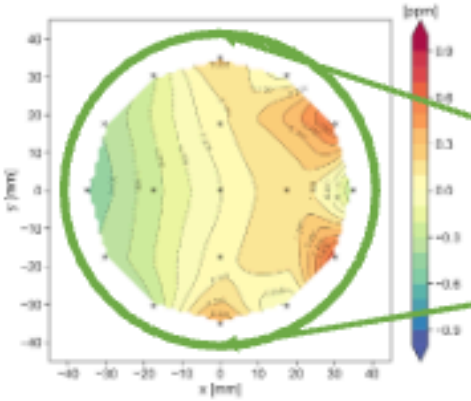
~ 0.025 ppm *B*-field measurement

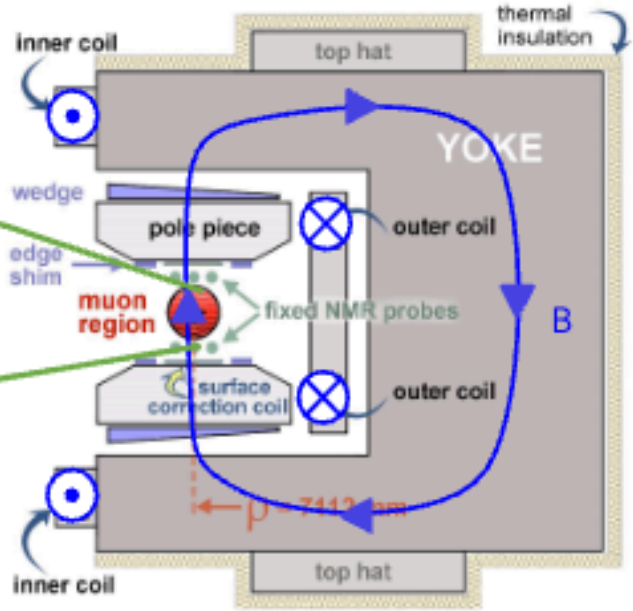
Use NMR trolley and probes: $|\vec{B}| = \omega_p/(2\mu_p)$

THE MAGNETIC FIELD



Field in muon storage region





- 1.45 T Magnet
- 14 m in diameter
- 12 iron yokes excited by superconducting coils

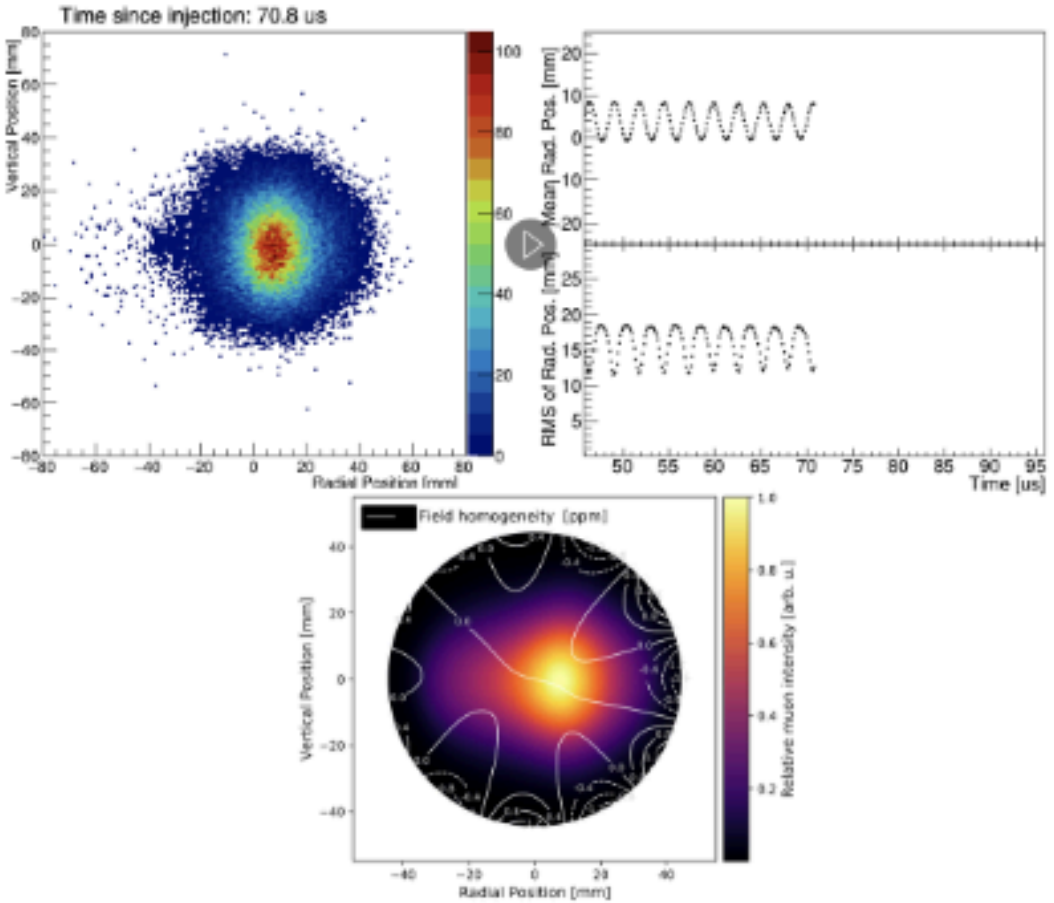
Edge shims and surface coils are used to shape the field

NMR probes (petroleum jelly) measure the proton Larmor frequency ($\omega_p \propto B$)

1. Fixed probes: 24/7
2. Trolley (in vacuum): every 3-4 days

Beam oscillations

TRACKERS



Trackers measure the beam oscillations directly this can help with:

Tuning simulations

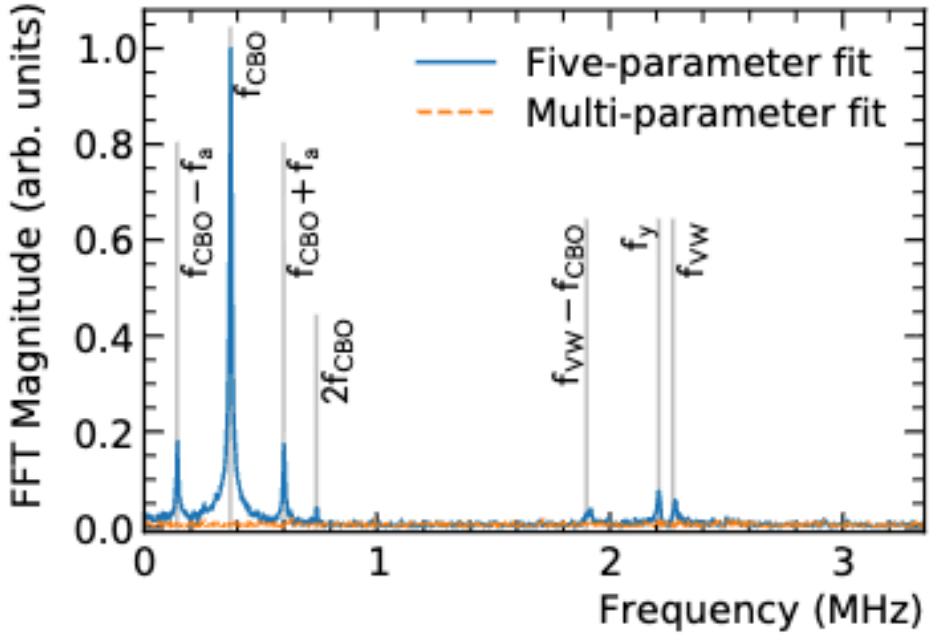
Beam dynamics corrections

Optimising experimental running

Weight the magnetic field maps where the muons are kept.

Fermilab E989 ω_a^m measurement

Real life is much more complicated . . .



ω_a^m shifted by 1.6 ppm !

Final fit functions have $>$ 25 parameters

$$
\begin{aligned}
N(t) = {} & N e^{-t/\tau_\mu}\left[1 + A\cdot\cos(\omega_a t - \phi + \phi_{BO}(t))\right]\cdot \\
& \cdot\left(1 + A_{CBO}\cos(\omega_{CBO}t - \phi_{CBO})e^{-\frac{t}{\tau_{CBO}}}\right)\cdot \\
& \cdot\left(1 + A_{VW}\cos(\omega_{VW}t - \phi_{VW})e^{-\frac{t}{\tau_{VW}}}\right)\cdot \\
& \cdot\left(1 + A_{2CBO}\cos(\omega_{2CBO}t - \phi_{2CBO})e^{-\frac{t}{\tau_{2CBO}}}\right)\cdot \\
& \cdot\left(1 + A_y\cos(\omega_y t - \phi_y)e^{-\frac{t}{\tau_y}}\right)\cdot \\
& \cdot\left(1 - k_{LM}\int_0^t L(t')e^{t'/\tau_\mu}dt'\right)\cdot \\
& \cdot\left(1 + [A_+\cos(\omega_+(t)t - \phi_+) + A_-\cos(\omega_-(t)t - \phi_-)]\,e^{-\frac{t}{\tau_{CBOVW}}}\right)
\end{aligned}
$$

From 5 parameters to 27 !

- accounts for beam loss
- for effects of beam oscillations
- for detector effects
- 5 analysis teams, 8 methods, 20 analyses
- blind analysis

Still not done !

$$a_\mu = \omega_a \frac{m_\mu}{e|\vec{B}|}, \qquad |\vec{B}| = \frac{\omega_p}{2\mu_p}, \qquad e = 2m_e\mu_B$$

MEASURING MUON G-2 - CORRECTIONS

There are seven small effects which need to be corrected for.

$\hbar\omega_p = 2\mu_p|\vec{B}|$

$$a_\mu = \frac{\omega_a}{\tilde{\omega}'_p(T_r)} \frac{\mu'_p(T_r)}{\mu_B} \frac{m_\mu}{m_e}$$

Unc. 4 ppb Unc. 22 ppb

Clock blinding frequency; Anomalous precession frequency; Beam dynamics corrections

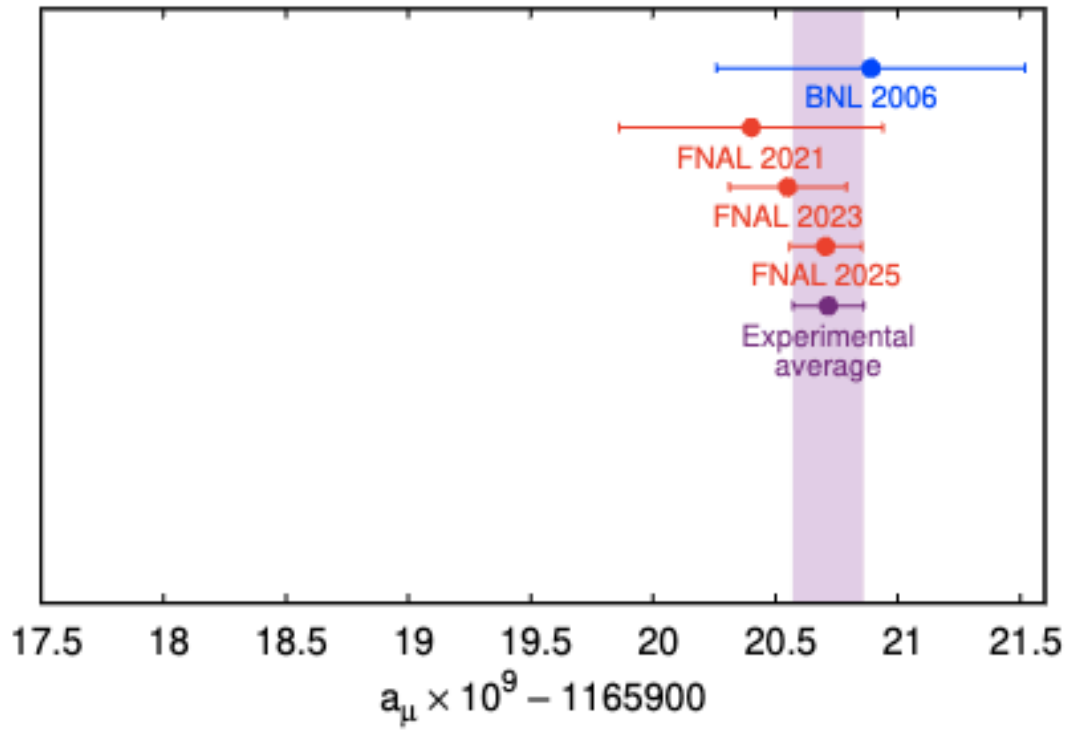
$$f_{clock}\,\omega_a^m\,(1 + C_e + C_p + C_{pa} + C_{dd} + C_{ml}) \Big/ f_{calib}\,\{\omega'_p(T_r) \times M\}(1 + B_k + B_q)$$

Absolute calibration frequency; Magnetic field map; Muon beam distribution; Transient magnetic field corrections

1. Quad Transient (B_q): vibrations of ESQ plates which disturb magnetic field. -37ppb
2. Kicker Transient (B_k): residual kicker eddy current. +21 ppb

1. Electric field correction (C_e): spread in momenta of muons. +347 ppb
2. Pitch Correction (C_p): vertical oscillation of muons. +175 ppb
3. Muon loss correction (C_{ml}): decay rate of muons not purely exponential. 0ppb
4. Phase acceptance correction (C_{pa}): energy dependence of positron drift time and time dependence of calorimeter acceptance. +33 ppb
5. Differential decay correction (C_{dd}): high-momentum muons have a longer lifetime. +26 ppb

a_μ: present experimental status



$$a_\mu|_{\text{expt}} = [11659207.15 \pm 1.45] \times 10^{-10} \qquad [0.124\,\text{ppm}]$$



vs



Standard model calculation of a_μ

[Aoyama et al '20 = WP '20, Aliberti et al '25 = WP '25]

At precision required by experiment, need all three interactions and all SM particles

$$
\begin{aligned}
a_\mu^{\mathrm{SM}} &= a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{had}} + a_\mu^{\mathrm{EW}} \\
&= O\left(\frac{\alpha}{2\pi}\right) + O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) + O\left(\left(\frac{\alpha}{16\pi \sin^2\theta_W}\right)\left(\frac{m_\mu}{M_W}\right)^2\right) \\
&= O\left(10^{-3}\right) + O\left(10^{-7}\right) + O\left(10^{-9}\right)
\end{aligned}
$$

QED contributions to a_ℓ

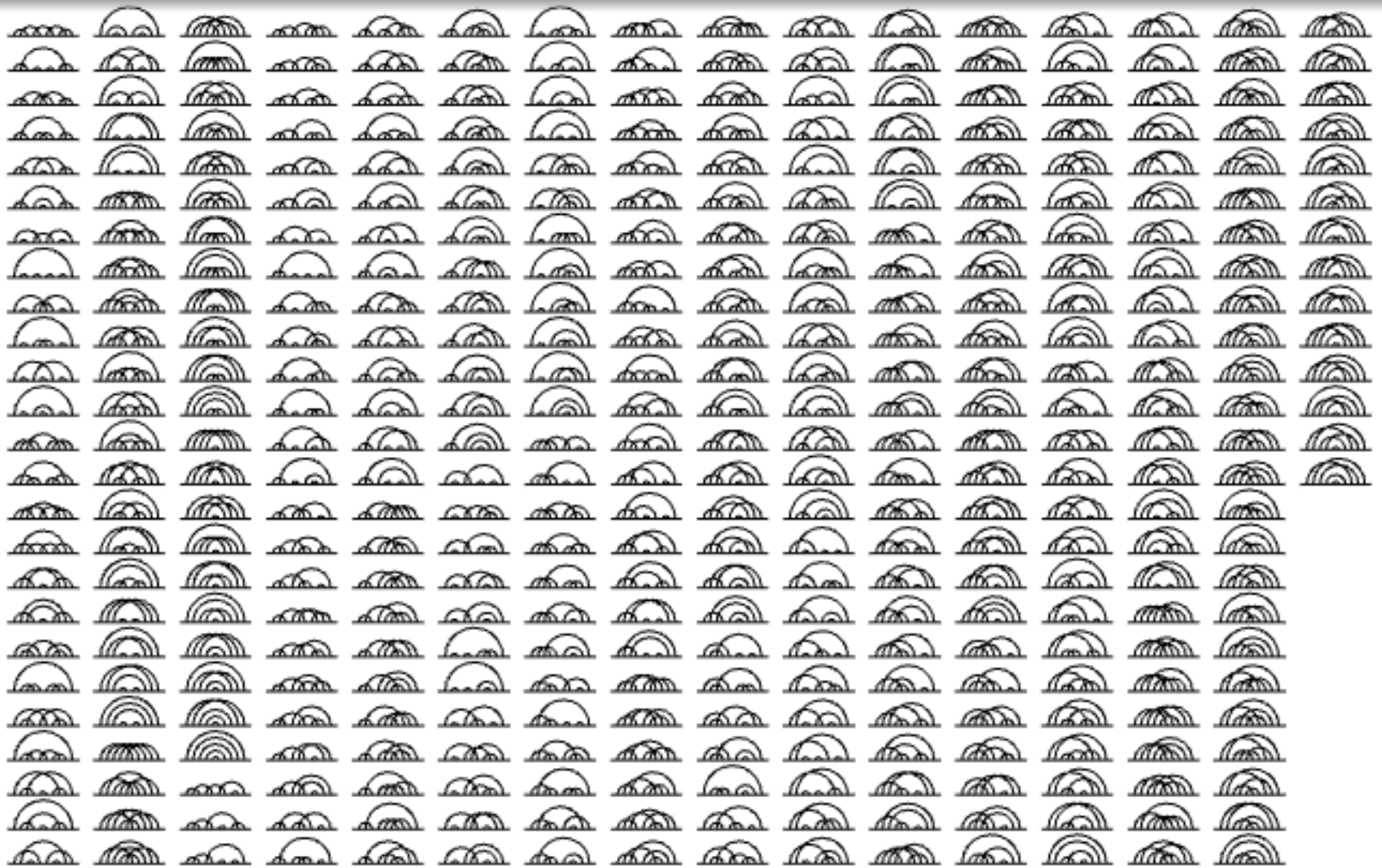
Loops with only photons and leptons: can expand in $\alpha = e^2/(4\pi) \ll 1$

$$a_\ell^{\rm QED} = C_\ell^{(2)} \left(\frac{\alpha}{\pi}\right) + C_\ell^{(4)} \left(\frac{\alpha}{\pi}\right)^2 + C_\ell^{(6)} \left(\frac{\alpha}{\pi}\right)^3 + C_\ell^{(8)} \left(\frac{\alpha}{\pi}\right)^4 + C_\ell^{(10)} \left(\frac{\alpha}{\pi}\right)^5 + \cdots$$

$$C_\ell^{(2n)} = A_1^{(2n)} + A_2^{(2n)}(m_\ell/m_{\ell'}) + A_3^{(2n)}(m_\ell/m_{\ell'}, m_\ell/m_{\ell''})$$

- $A_1^{(2)}, A_1^{(4)}, A_1^{(6)}, A_2^{(4)}, A_2^{(6)}, A_3^{(6)}$ known analytically [Schwinger '48; Sommerfield '57, '58; Petermann '57; ...]
- $O((\alpha/\pi)^3)$: 72 diagrams [Laporta et al '91, '93, '95, '96; Kinoshita '95)
- $O((\alpha/\pi)^4; (\alpha/\pi)^5)$: 891;12,672 diagrams [Laporta '95; Aguilar et al '08; Aoyama et al '96-'19, Volkov '19-'24]
  - Automated generation of diagrams
  - Numerical evaluation of loop integrals
  - Calculations cross-checked

5-loop QED master integrals



[Aoyama et al '15]

$$\rightarrow a_\ell^{\mathrm{QED}} = C_\ell^{(2)} \left(\frac{\alpha}{\pi}\right) + C_\ell^{(4)} \left(\frac{\alpha}{\pi}\right)^2 + C_\ell^{(6)} \left(\frac{\alpha}{\pi}\right)^3 + C_\ell^{(8)} \left(\frac{\alpha}{\pi}\right)^4 + C_\ell^{(10)} \left(\frac{\alpha}{\pi}\right)^5 + \cdots$$

QED contribution to a_μ

Then w/ very precise value of α (e.g. α[Cs]):

					% of a_μ	order
$a_\mu^{\rm QED} \times 10^{10}$	$=$		11 614 097.3321	(23)	99.6133%	α
		$+$	41 321.7626	(7)	0.3544%	α^2
		$+$	3 014.1902	(33)	0.0259%	α^3
		$+$	38.1004	(17)	0.0003%	α^4
[needs negligible update]		$+$	0.5078	(6)	$4 \cdot 10^{-6}$	α^5
	$=$		11 658 471.8931	$(7)_{m_\tau}(17)_{\alpha^4}(6)_{\alpha^5}(100)_{\alpha^6}(23)_\alpha[104]$	[0.9 ppb]	

[Aoyama et al '12, '18, '19, '25; Volkov '19, '24]

Conservatively combine results w/ α[Cs], α[Rb], $\alpha[a_e]$

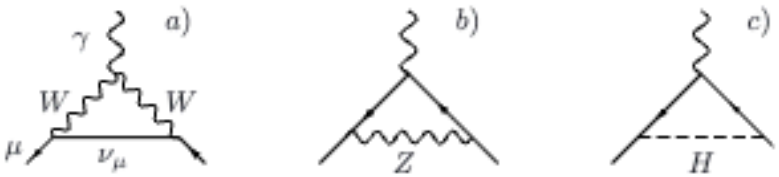
$$\xrightarrow{\text{[WP '25]}} a_\mu^{\rm QED} \times 10^{10} = 11\,658\,471.88(2)\ [1.7\ \text{ppb}]$$

99.994% of a_μ are due to QED contributions!

$$a_\mu^{\rm exp} - a_\mu^{\rm QED} = 735.3(1.5) \times 10^{-10}$$
$$\stackrel{?}{=} a_\mu^{\rm EW} + a_\mu^{\rm had}$$

Electroweak contributions to a_μ: Z, W, H, etc. loops

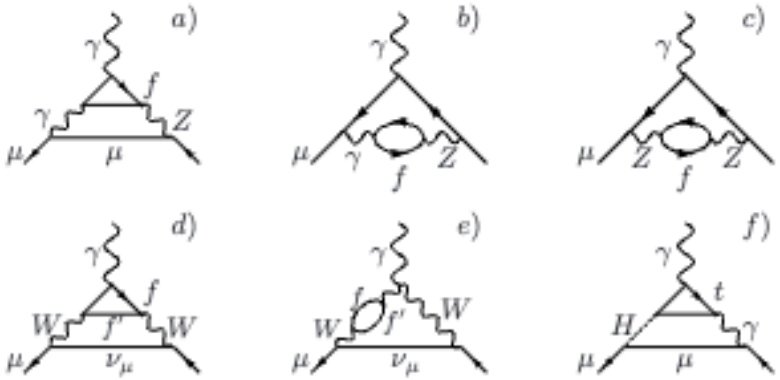
1-loop



$$a_\mu^{\mathrm{EW},(1)} = O\left(\frac{\sqrt{2}G_F m_\mu^2}{16\pi^2}\right) = 19.479(1)\times 10^{-10}$$

[Gnendiger et al '15, Aliberti et al '20 and refs therein]

2-loop



$$a_\mu^{\mathrm{EW},(2)} = O\left(\frac{\sqrt{2}G_F m_\mu^2}{16\pi^2}\frac{\alpha}{\pi}\right) = -4.038(30)\times 10^{-10}$$

[Gnendiger et al '15, Aliberti et al '20 and refs therein]

$$a_\mu^{\mathrm{EW}} = 15.44(4)\times 10^{-10}$$

Hadronic contributions to a_μ: quark and gluon loops

$$a_\mu^{\text{exp}} - a_\mu^{\text{QED}} - a_\mu^{\text{EW}} = 719.8(1.5) \times 10^{-10} \stackrel{?}{=} a_\mu^{\text{had}}$$

- Clearly right order of magnitude:

$$a_\mu^{\text{had}} = O\left(\left(\frac{\alpha}{\pi}\right)^2 \left(\frac{m_\mu}{M_\rho}\right)^2\right) = O\left(10^{-7}\right)$$

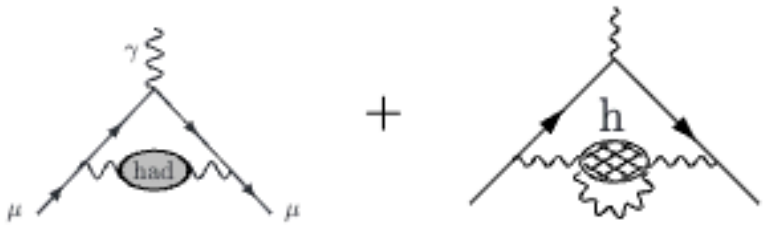
(already Gourdin & de Rafael '69 found $a_\mu^{\text{had}} = 650(50) \times 10^{-10}$)

- However, must be determined to subpercent accuracy & involves quarks and gluons at low energies
  - $\Rightarrow$ must be able to describe the highly nonlinear dynamics of the strong interaction in that regime
  - $\Rightarrow$ cannot rely on the perturbative methods used for QED and weak corrections
  - $\Rightarrow$ need methods that allow a fully nonperturbative calculation

- Decompose:

$$a_\mu^{\text{had}} = a_\mu^{\text{LO-HVP}} + a_\mu^{\text{HO-HVP}} + a_\mu^{\text{HLbyL}} + O\left(\left(\frac{\alpha}{\pi}\right)^4\right)$$

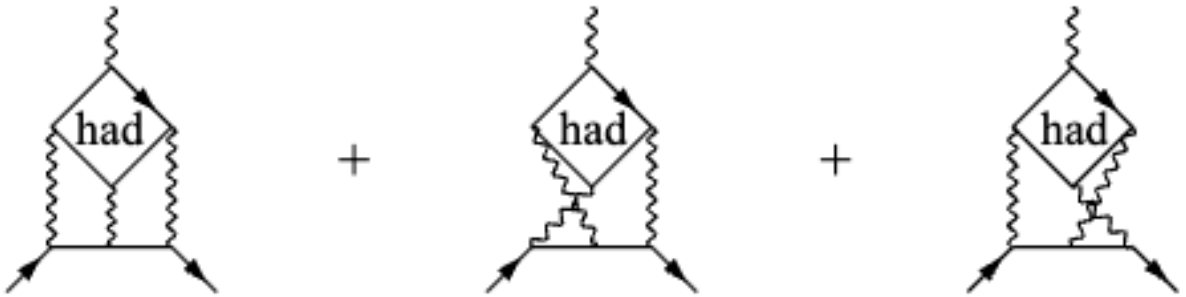
Hadronic contributions to a_μ: diagrams



$$\rightarrow a_\mu^{\text{LO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^2\right)$$



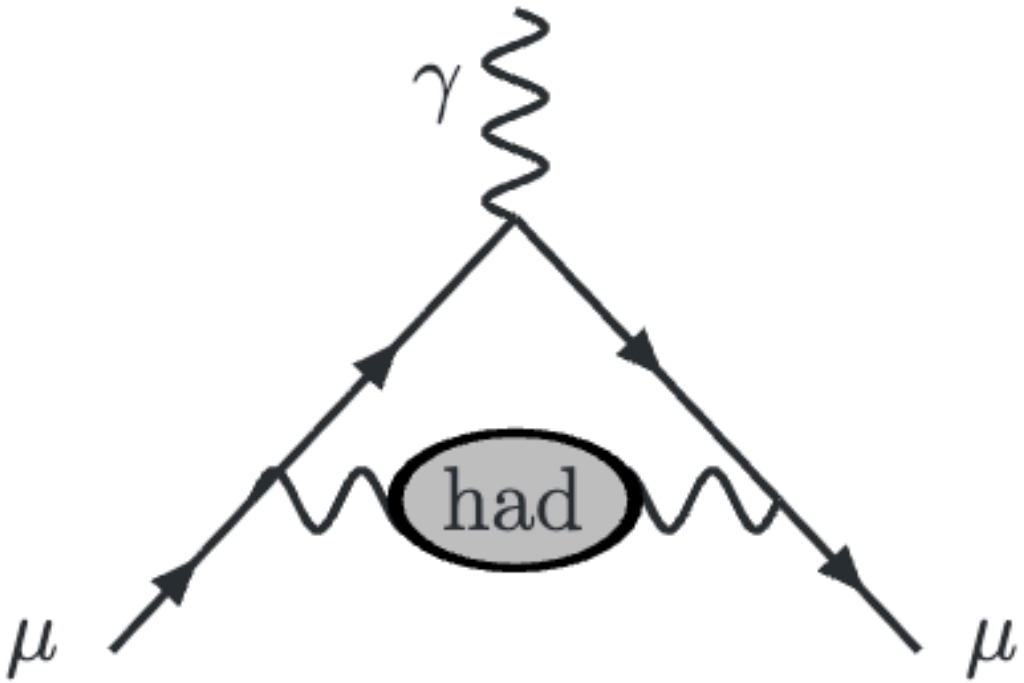
$$\rightarrow a_\mu^{\text{NLO-HVP}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$



$$\rightarrow a_\mu^{\text{HLbL}} = O\left(\left(\frac{\alpha}{\pi}\right)^3\right)$$

Data-driven determination of leading-order hadonic vacuum polarization contribution (LO-HVP) to a_μ

Only approach available until (relatively) recently



$a_\mu^{\text{LO-HVP}}$: introduction

Consider HVP for spacelike momenta (i.e. in Euclidean spacetime) [Blum '02]

$$\Pi_{\mu\nu}(Q) =$$



$$= \int d^4x\, e^{iQ\cdot x} \langle J_\mu(x) J_\nu(0)\rangle$$

$$= \left(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2\right) \Pi(Q^2)$$

w/ $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots$

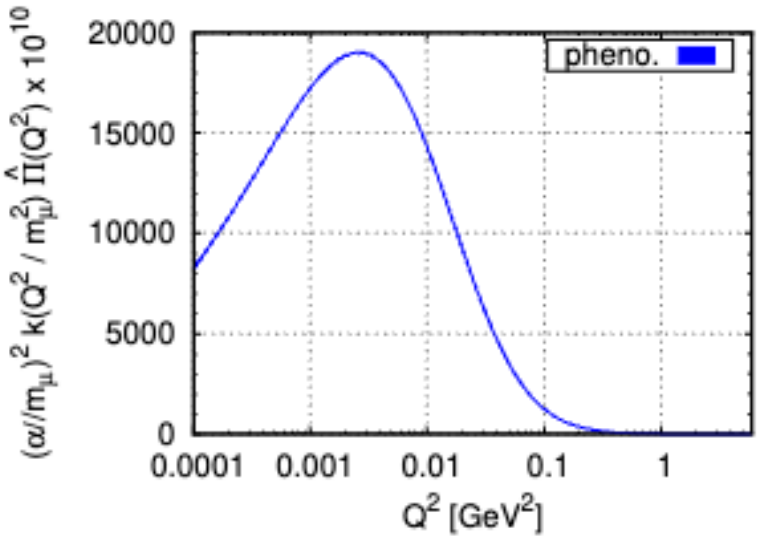
Then [Lautrup et al '69, Blum '02]

$$a_\ell^{\text{LO-HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{dQ^2}{m_\ell^2}\, w(Q^2/m_\ell^2)\hat{\Pi}(Q^2)$$

w/ $\hat{\Pi}(Q^2) \equiv \left[\Pi(Q^2) - \Pi(0)\right]$

Integrand peaked for $Q \sim (m_\ell/2) \sim 50\,\text{MeV}$ for μ

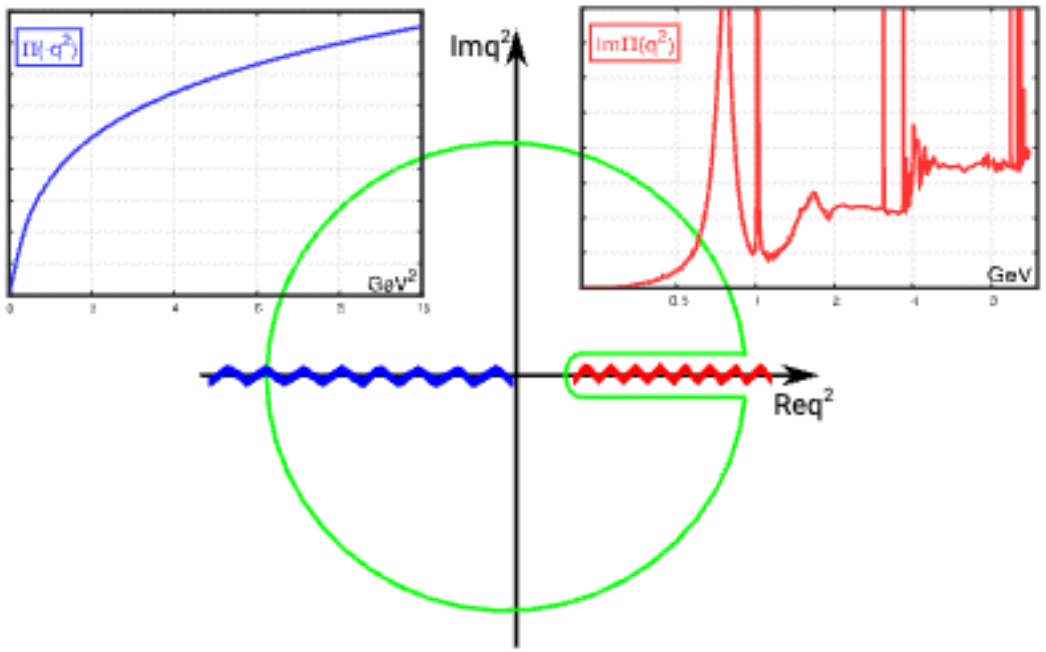
$\Rightarrow$ must contend with nonperturbative strong-interaction dynamics



[HVP from Jegerlehner, "alphaQEDc17" (2017)]

$a_\mu^{\text{LO-HVP}}$: dispersive framework [Bouchiat et al '61]

$\hat{\Pi}(q^2)$ is a real analytic function except for a cut along real, positive q^2 axis

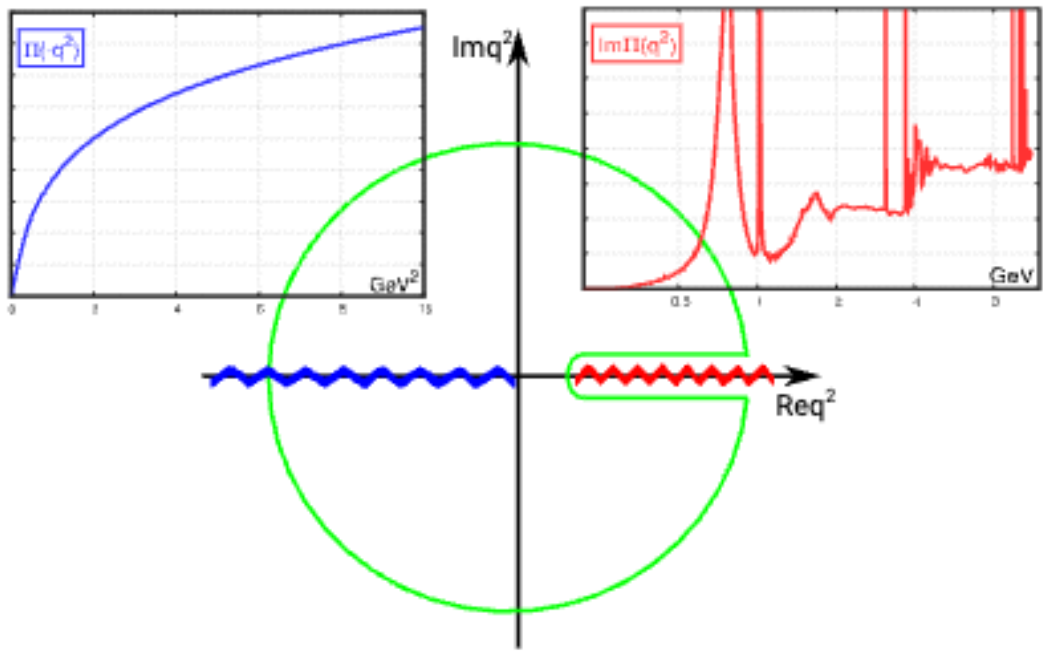


- **Analyticity & $|q^2| \to \infty$ aymptotics**

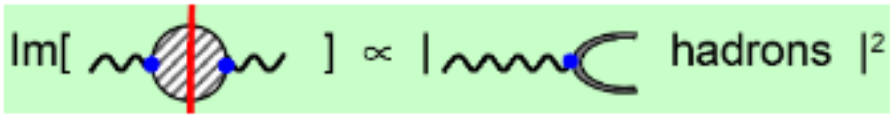
$$\hat{\Pi}(q^2 = s) = \Pi(s) - \Pi(0) = \frac{s}{\pi} \int_{s_{\text{th}}}^{\infty} ds' \, \frac{\text{Im}\Pi(s')}{s'(s' - s - i\epsilon)}$$

$a_\mu^{\text{LO-HVP}}$: dispersive framework [Bouchiat et al '61]

$\hat{\Pi}(q^2)$ is a real analytic function except for a cut along real, positive q^2 axis



- **Unitarity** (optical theorem)



$$\mathrm{Im}\Pi(q^2 = s) \propto \sigma(e^+e^-(s) \to \gamma^* \to \text{had})$$

$a_\mu^{\text{LO-HVP}}$: data-driven evaluation

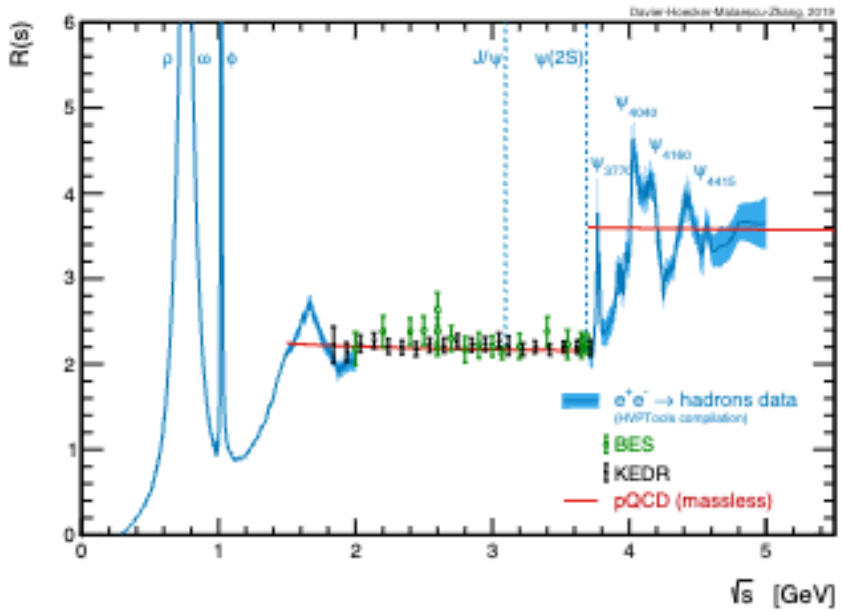
- Use R-ratio

$$R(s) = \frac{\sigma(e^+e^- \to \gamma^* \to \text{hadrons})}{(4\pi\alpha(s)^2/3s)}$$

- Combine expressions above ($\widehat{K}(s)$, known kernel)

$$a_\mu^{\text{LO-HVP}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \left\{ \int_{m_\pi^2}^{s_{\text{pQCD}}} \frac{ds}{s^2} \widehat{K}(s) R_{\text{data}}(s) + \int_{s_{\text{pQCD}}}^{\infty} \frac{ds}{s^2} \widehat{K}(s) R_{\text{pQCD}}(s) \right\}$$

- Integrate data compiled from many experiments & pQCD to $O(\alpha_s^4)$ [from Davier et al, DHMZ '19]



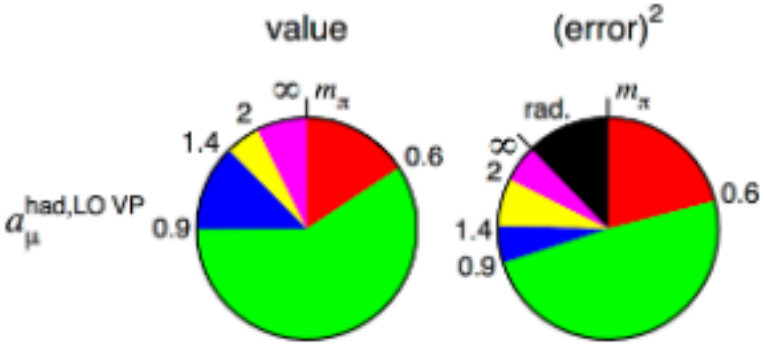
Data-driven channel contributions ca. 2020

- Many experiments & > 30 channels measured over decades
- Experiments cover only a certain $\sqrt{s}$ range
  $\rightarrow$ must combine, accounting for correlations
- Today, 2 main groups: Davier et al [DHMZ '19] & Keshavarzi et al [KNTW '20]

Channel	$a_\mu^{\mathrm{had,LO}}\ [10^{-10}]$
$\pi^0\gamma$	$4.29 \pm 0.06 \pm 0.04 \pm 0.07$
$\eta\gamma$	$0.65 \pm 0.02 \pm 0.01 \pm 0.01$
$\pi^+\pi^-$	$507.80 \pm 0.83 \pm 3.19 \pm 0.60$
$\pi^+\pi^-\pi^0$	$46.20 \pm 0.40 \pm 1.10 \pm 0.86$
$2\pi^+2\pi^-$	$13.68 \pm 0.03 \pm 0.27 \pm 0.14$
$\pi^+\pi^-2\pi^0$	$18.03 \pm 0.06 \pm 0.48 \pm 0.26$
$2\pi^+2\pi^-\pi^0$ (η excl.)	$0.69 \pm 0.04 \pm 0.06 \pm 0.03$
$\pi^+\pi^-3\pi^0$ (η excl.)	$0.49 \pm 0.03 \pm 0.09 \pm 0.00$
$3\pi^+3\pi^-$	$0.11 \pm 0.00 \pm 0.01 \pm 0.00$
$2\pi^+2\pi^-2\pi^0$ (η excl.)	$0.71 \pm 0.06 \pm 0.07 \pm 0.14$
$\pi^+\pi^-4\pi^0$ (η excl., isospin)	$0.08 + 0.01 + 0.08 + 0.00$
$\eta\pi^+\pi^-$	$1.19 + 0.02 + 0.04 + 0.02$
$\eta\omega$	$0.35 + 0.01 + 0.02 + 0.01$
$\eta\pi^+\pi^-\pi^0$(non-ω,ϕ)	$0.34 + 0.03 + 0.03 + 0.04$
$\eta 2\pi^+2\pi^-$	$0.02 + 0.01 + 0.00 + 0.00$
$\omega\eta\pi^0$	$0.06 + 0.01 + 0.01 + 0.00$
$\omega\pi^0$ ($\omega \to \pi^0\gamma$)	$0.94 + 0.01 + 0.03 + 0.00$
$\omega 2\pi$ ($\omega \to \pi^0\gamma$)	$0.07 + 0.00 + 0.00 + 0.00$
ω (non $3\pi, \pi\gamma, \eta\gamma$)	$0.04 \perp 0.00 \perp 0.00 \perp 0.00$
K^+K^-	$23.08 \perp 0.20 \perp 0.33 \perp 0.21$
$K_S K_L$	$12.82 \perp 0.06 \perp 0.18 \perp 0.15$
ϕ (non $K\overline{K}, 3\pi, \pi\gamma, \eta\gamma$)	$0.05 \perp 0.00 \perp 0.00 \perp 0.00$
$K\overline{K}\pi$	$2.45 \perp 0.05 \perp 0.10 \perp 0.06$
$K\overline{K}2\pi$	$0.85 \perp 0.02 \perp 0.05 \perp 0.01$
$K\overline{K}3\pi$ (estimate)	$-0.02 \perp 0.01 \perp 0.01 \perp 0.00$
$\eta\phi$	$0.33 \perp 0.01 \perp 0.01 \perp 0.00$
$\eta K\overline{K}$ (non-ϕ)	$0.01 + 0.01 + 0.01 + 0.00$
$\omega K\overline{K}$ ($\omega \to \pi^0\gamma$)	$0.01 + 0.00 + 0.00 + 0.00$
$\omega 3\pi$ ($\omega \to \pi^0\gamma$)	$0.06 + 0.01 + 0.01 + 0.01$
7π ($3\pi^+3\pi^-\pi^0$ + estimate)	$0.02 + 0.00 + 0.01 + 0.00$
J/ψ (BW integral)	$6.28 \perp 0.07$
$\psi(2S)$ (BW integral)	$1.57 \perp 0.03$
R data $[3.7 - 5.0]$ GeV	$7.29 \perp 0.05 \perp 0.30 \perp 0.00$
$R_{\mathrm{QCD}}\,[1.8 - 3.7\ \mathrm{GeV}]_{\mathrm{uds}}$	$33.45 \pm 0.28 \pm 0.65_{\mathrm{stat}}$
$R_{\mathrm{QCD}}\,[5.0 - 9.3\ \mathrm{GeV}]_{\mathrm{udsc}}$	6.86 ± 0.04
$R_{\mathrm{QCD}}\,[9.3 - 12.0\ \mathrm{GeV}]_{\mathrm{udscb}}$	1.21 ± 0.01
$R_{\mathrm{QCD}}\,[12.0 - 40.0\ \mathrm{GeV}]_{\mathrm{udscb}}$	1.64 ± 0.00
$R_{\mathrm{QCD}}\,[> 40.0\ \mathrm{GeV}]_{\mathrm{udscb}}$	0.16 ± 0.00
$R_{\mathrm{QCD}}\,[> 40.0\ \mathrm{GeV}]_{\mathrm{t}}$	0.00 ± 0.00
Sum	$693.9 \pm 1.0 \pm 3.4 \pm 1.6 \pm 0.1_{\psi} \pm 0.7_{\mathrm{QCD}}$

Data-driven channel contributions ca. 2020

- Many experiments & > 30 channels measured over decades
- Experiments cover only a certain $\sqrt{s}$ range
  $\rightarrow$ must combine, accounting for correlations
- Today, 2 main groups: Davier et al [DHMZ '19] & Keshavarzi et al [KNTW '20]



$\gtrsim 70\%$ of $a_\mu^{\text{LO-HVP}}$ comes from $\pi^+\pi^-$ (ρ resonance)

2020 White Paper of the muon $g-2$ Theory Initiative [WP '20]
$\rightarrow$ lattice not yet competitive so use data-driven only

$$a_\mu^{\text{LO-HVP}} = 693.1(2.8)_{\text{exp}}(0.7)_{\text{DV+QCD}}(2.8)_{\text{KLOE/BABAR}} \times 10^{-10} \quad [0.58\%]$$

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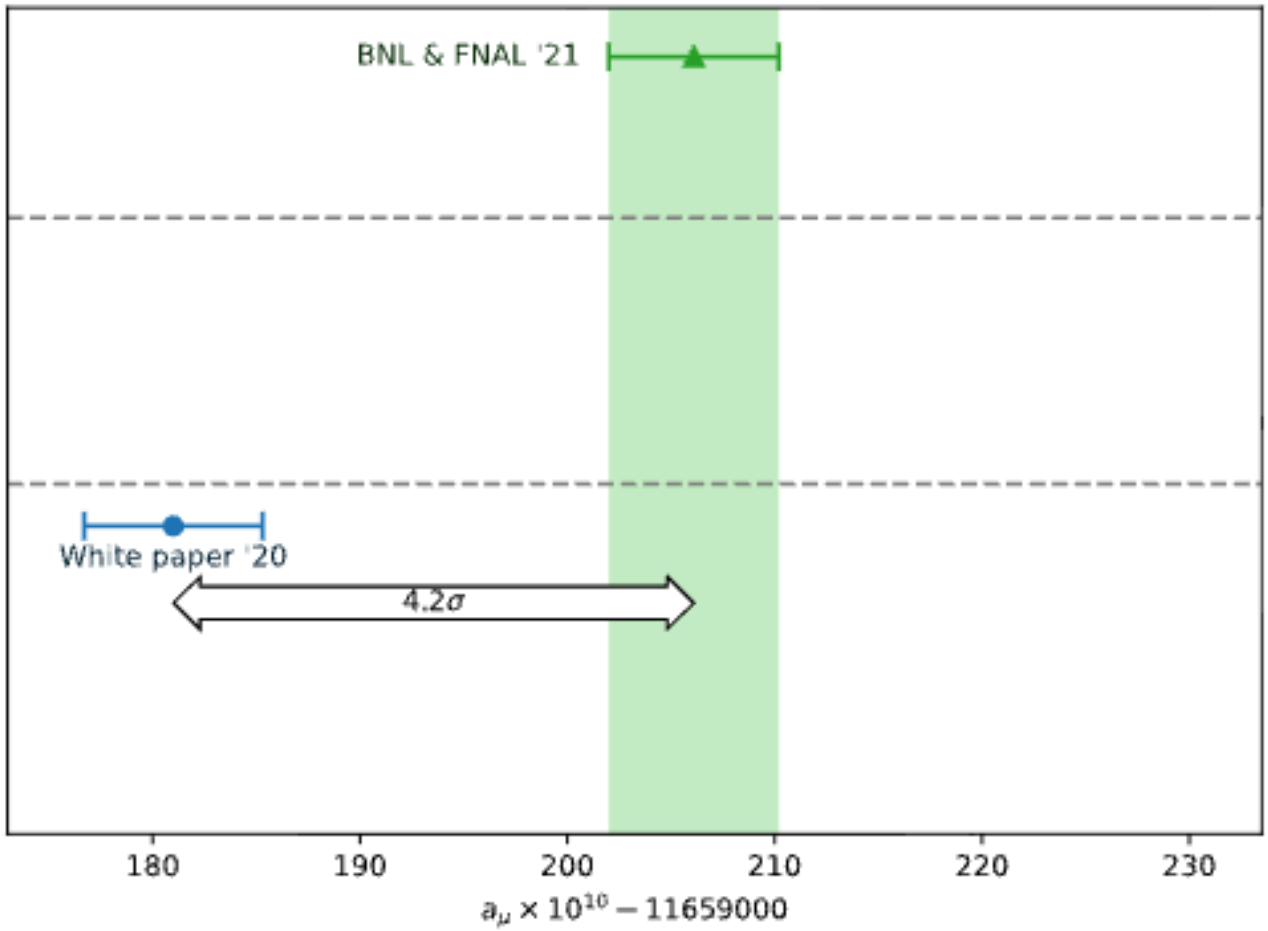
2020 Standard Model predictions and comparison to experiment

[WP '20, BMW '20]

Experiment vs SM: April 2021



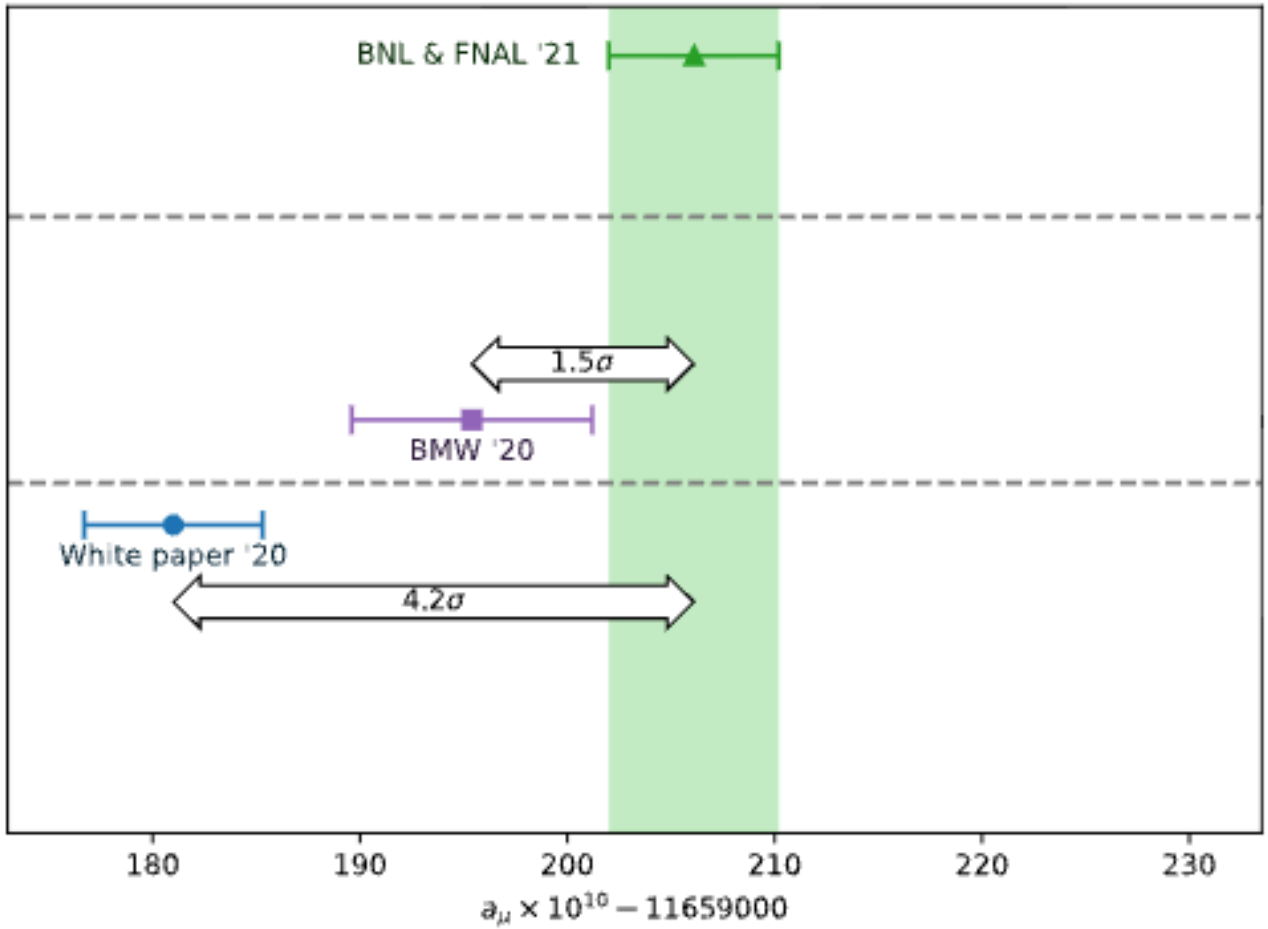
Concensus: strong evidence for new fundamental physics

However, error on $a_\mu^{\text{LO-HVP}}$ is 87% of total squared error on a_μ in WP '20

$\Rightarrow$ need fully independent calculation of $a_\mu^{\text{LO-HVP}}$

$\rightarrow$ **Lattice QCD**

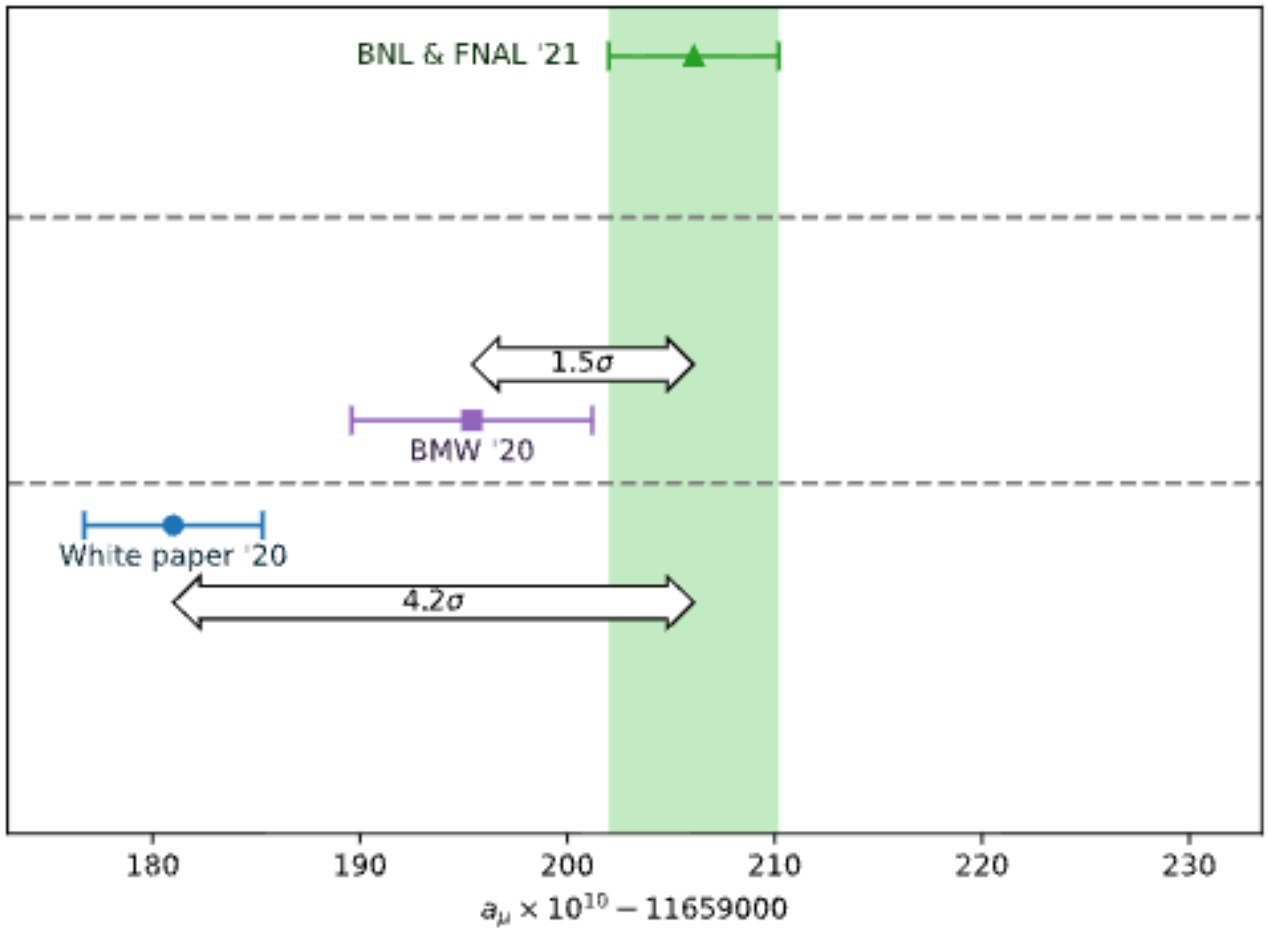
Experiment vs SM: April 2021



Overall picture changes:

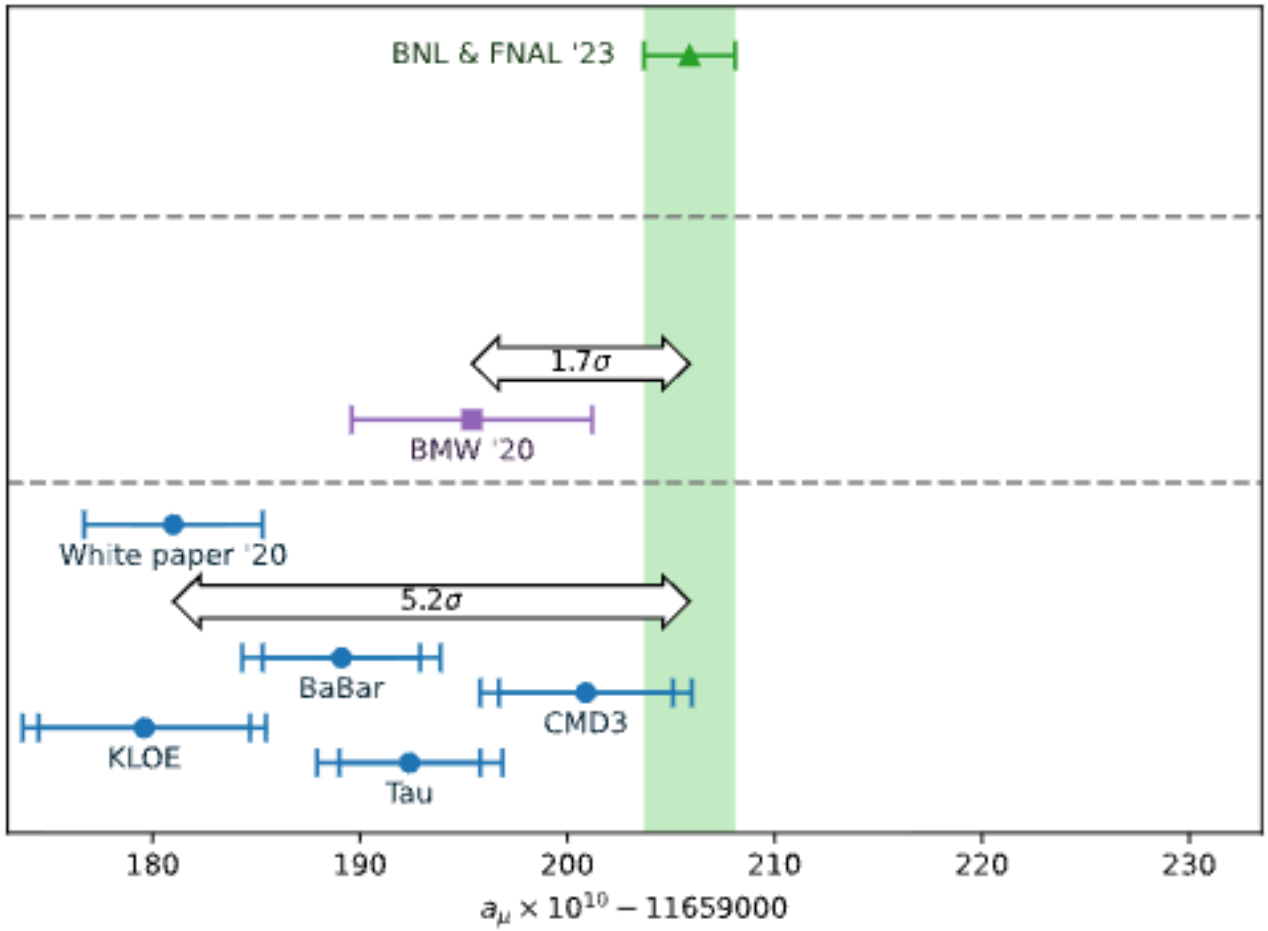
- Complete lattice calculation of HVP contribution in BMW '20 suggests SM still in the race
- Computation of sub-contribution in BMW '20 shows 3.7σ tension between lattice and data-driven: both cannot be correct

Experiment vs SM: April 2021



"if the result of the present [BMW work] is right, when taken at face value, the consequences for particle physics would be quite dramatic. [. . .] several different and independent experiments [. . .] would have to be all wrong in a fundamental way." [an eminent colleague]

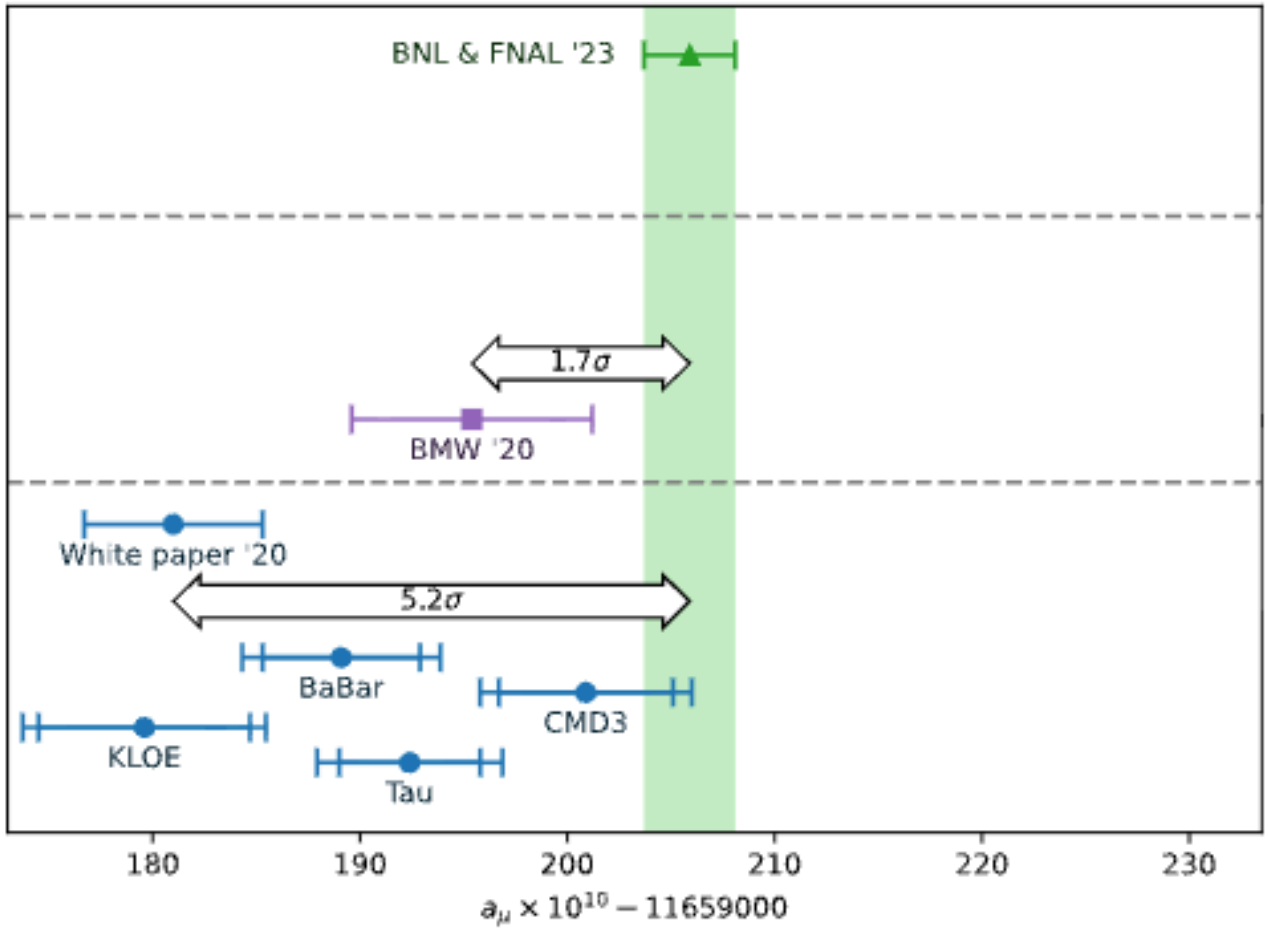
Experiment vs SM: 2023



New measurement of $\sigma(e^+e^- \to \pi^+\pi^-)$ in CMD-3 '23 further challenges WP '20 data-driven consensus

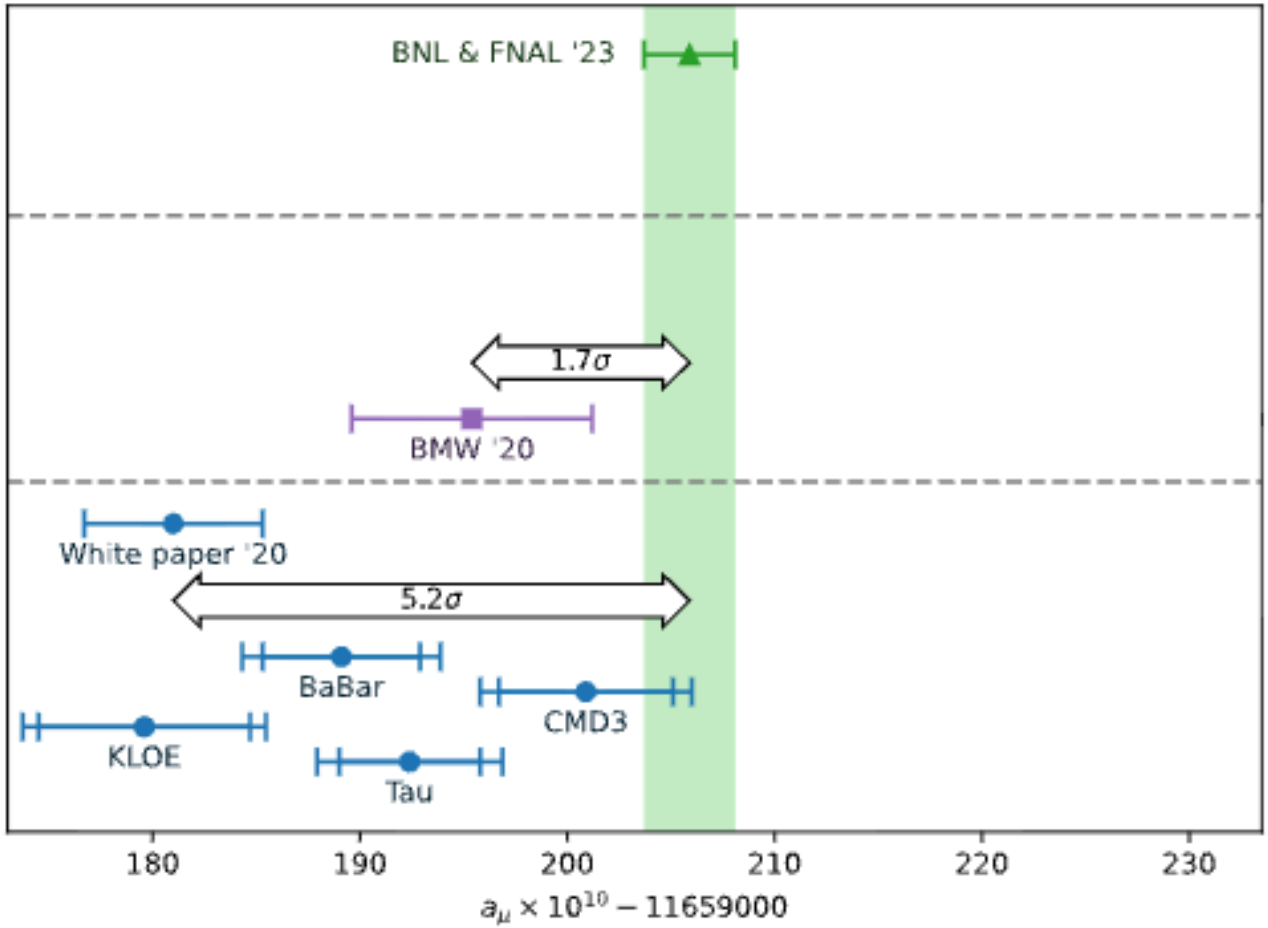
- → resulting $a_\mu^{\text{LO-HVP}}$ agrees w/ BMW '20 one to w/in 1.5σ
- → re-assessment of data-driven results further challenges consensus [DHLMZ '23]

Experiment vs SM: 2023



Must do better $\longrightarrow$ new BMW-DMZ calculation

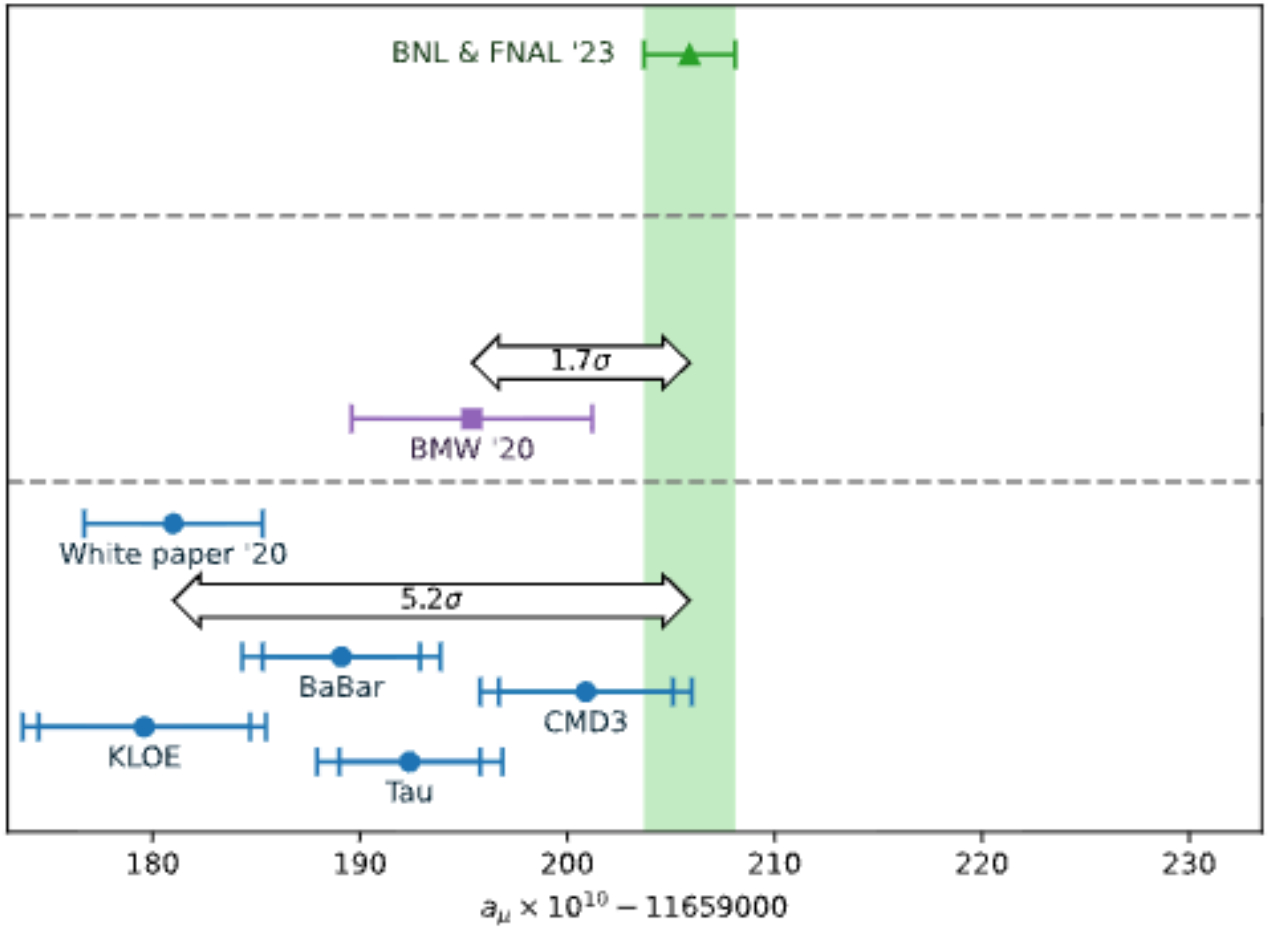
Experiment vs SM: 2023



Must do better ⟶ new BMW-DMZ calculation

A lot of lattice QCD

Experiment vs SM: 2023



Must do better ⟶ new BMW-DMZ calculation

A lot of lattice QCD w/ a pinch of e^+e^-

Brief introduction to lattice QCD

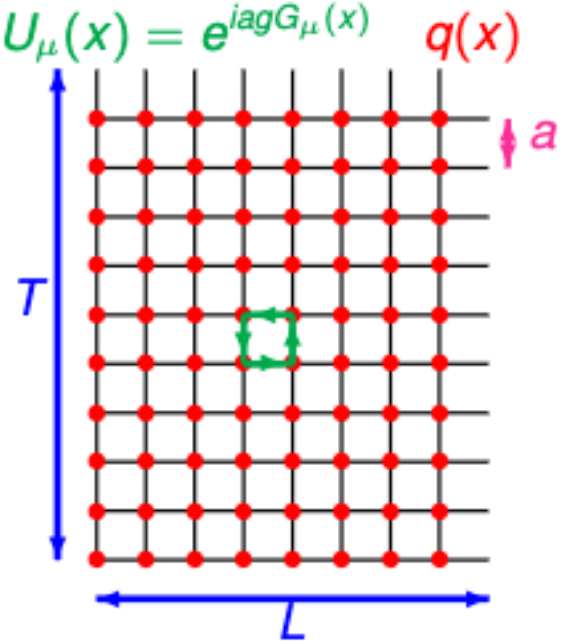
Lattice QCD (LQCD): introduction

- First-principle calculation: evaluation of QCD path integral in *Euclidean spacetime*, $iS \to -S_E$

$$\langle O \rangle = \int \mathcal{D}U\mathcal{D}\bar{q}\mathcal{D}q\, e^{-S_E[U,\bar{q},q]}\, O[q,\bar{q},U]$$

- Regularize by defining theory on a hypercubic lattice
  - $a \to$ UV regulator
    $\to$ typically, $a \sim 0.1\,\text{fm} \to 0.04\,\text{fm}$
  - $T \times L^3$
    $\to$ typically $L \sim 6\,\text{fm}\,[\to 12\,\text{fm}]$
    $\to$ up to $L/a \sim 128$

$\Rightarrow$ well-defined functional integral



- Discretized action not unique $\to$ Wilson, Lüscher-Weisz, Staggered, Domain Wall, ...
- ... but all must have QCD as continuum limit

$$S_E^{\text{lat}} \xrightarrow{a\to 0} \frac{1}{2g_0^2}\int \text{Tr}\, G_{\mu\nu}G_{\mu\nu} + \sum_{q=u,d,s,c,\ldots}\int \bar{q}(\gamma_\mu D_\mu + m_q)q$$

LQCD: computing the lattice path integral

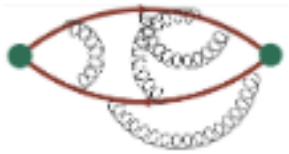
$$\langle O \rangle = \int \mathcal{D}U A \mathcal{D}\bar{q} \mathcal{D}q \, e^{-S_G[U] - \int \bar{q} D[M,U] q} \, O[q, \bar{q}, U]$$

$$= \int \mathcal{D}U \, e^{-S_G[U]} \det(D[M,U]) \, O[U]_{\text{Wick}}$$

Example:

$$O[q(=u,d), \bar{q}, U] = [\bar{d}\gamma_\mu u](x) \, [\bar{u}\gamma_\nu d](y)$$

$$\xrightarrow{\text{Wick}} \quad O[U]_{\text{Wick}} = -\,\text{Tr}[S_d[y,x;U]\gamma_\mu S_u[x,y;U]\gamma_\nu] \quad =$$



Integral in finite # of dimensions & $\mathcal{D}U \, e^{-S_G[U]} \det(D[M,U]) \geq 0$
$\rightarrow$ stochastic evaluation using Hybrid Monte-Carlo (HMC)

- generate N gauge configurations $\{U_\mu\}_{i=1,\cdot,N}$ w/ weight $e^{-S_G[U]} \det(D[M,U]) \geq 0$

$$\{U_\mu\}_1 \rightarrow \{U_\mu\}_2 \rightarrow \{U_\mu\}_3 \rightarrow \cdots \rightarrow \{U_\mu\}_N$$

- compute $O[\{U_\mu\}_i]$ for $i = 1, \cdots, N$

$$\overline{O} = \frac{1}{N} \sum_{i=1}^{N} O[U_\mu^{(i)}]_{\text{Wick}} = \langle O \rangle + \delta O \;\; [\rightarrow \text{ statistical error}]$$

From LQCD to QCD

LQCD is just a regularization of QCD

- Parameters are dimensionless, bare QCD parameters: no more, no less

$$g_0,\ \tilde{m}_{ud},\ \tilde{m}_s,\ \tilde{m}_c$$

except for # of sites N_s along a side of hypercube eventually taken to ∞

$\rightarrow$ observables are dimensionless ratios, e.g.

$$[\tilde{M}_\pi/\tilde{M}_\Omega](g_0, \tilde{m}_{ud}, \cdots)$$

- For fixed g_0 ($N_s \to \infty$), tune $\tilde{m}_{ud}, \cdots$ to reproduce Φ values of as many dimensionless observables as quark masses (renormalization), e.g.

$$[\tilde{M}_\pi/\tilde{M}_\Omega](g_0, \tilde{m}^\Phi_{ud}, \cdots) = (M_\pi/M_\Omega)^\Phi,\ [\tilde{M}_K/\tilde{M}_\Omega](g_0, \tilde{m}^\Phi_{ud}, \cdots) = (M_K/M_\Omega)^\Phi,$$

$$[\tilde{M}_{D_s}/\tilde{M}_\Omega](g_0, \tilde{m}^\Phi_{ud}, \cdots) = (M_{D_s}/M_\Omega)^\Phi$$

$\rightarrow \tilde{m}^\Phi_{ud}, \cdots$ are functions of g_0

- Then compute any other observable of interest, e.g.

$$a_\mu^{\text{LO-HVP}}(g_0, \tilde{m}^\Phi_{ud}(g_0) \cdots)$$

From LQCD to QCD

- Connecting to the world of eVs and the continuum limit

$$\tilde{M}_\Omega^\Phi(g_0) = aM_\Omega^\Phi \Rightarrow a(g_0) = \tilde{M}_\Omega^\Phi(g_0)/M_\Omega^\Phi$$

$\Rightarrow$ Recover full continuum QCD result via asymptotic freedom ($\lim_{g_0 \to 0} a(g_0) = 0$)

$$a_\mu^{\text{LO-HVP}} = \lim_{a \to 0} \lim_{L \to \infty} a_\mu^{\text{LO-HVP}}(a, \tilde{m}_{ud}^\Phi(a) \cdots, L = aN_s)$$

which exists because $g_0 \to 0$ is a 2nd-order phase transition fixed point

Once done

$$\text{Lattice QCD} = \text{QCD}$$

HUGE conceptual, algorithmic and numerical challenge:

- path integrals in $O(10^{10} - 10^{11})$ real dimensions
- solving of linear systems w/ $O(10^{10})$ variables ...
- ... 1000s of times on 1000s of gauge configurations ...
- ... for a number of m_q, $q = u, d, s, c$ around their physical values ...
- ... and a number of a and L

Our particle "accelerators"

Such computations require some of the world's most powerful supercomputers



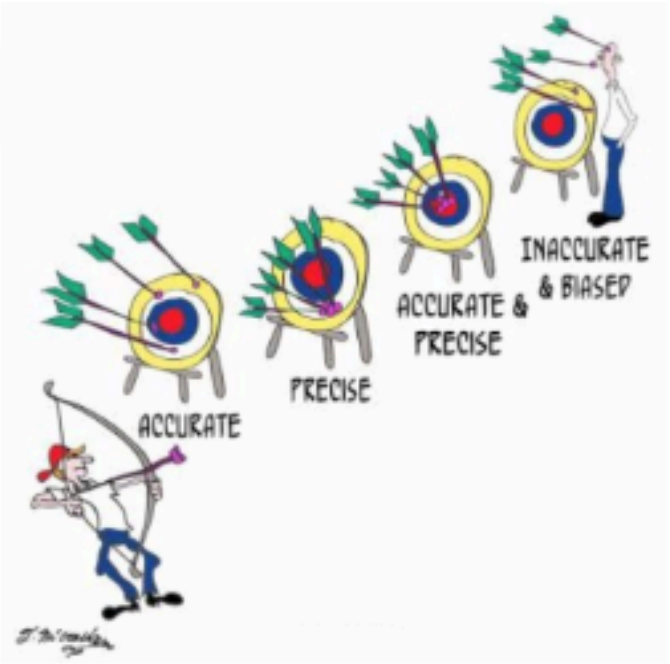




- LUMI: up to $\sim 4 \times 10^{17}$ flop/s
- Now in USA, Europe and Asia: exaflop supercomputers ($> 10^{18}$ flop/s)
- Many many thanks to GENCI, GCS, EuroHPC and the European taxpayer

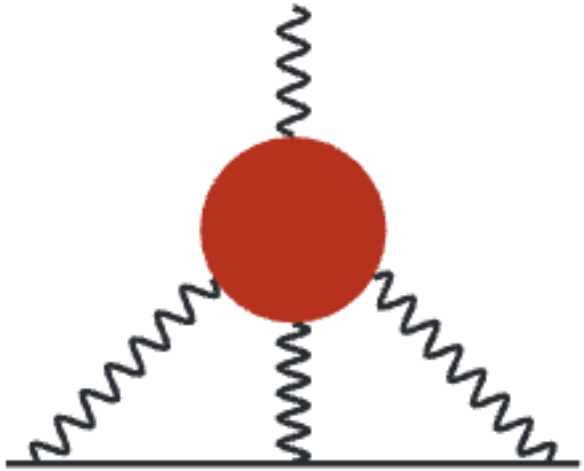
Lattice QCD: sources of uncertainty

- Statistical uncertainties
  - Monte-Carlo $\to 1/\sqrt{N_{\text{meas}}}$
- Systematic uncertainties
  - continuum extrapolation ($a \to 0$, typical $a \sim 0.10 \to 0.04\,\text{fm}$)
  - infinite-volume extrapolation ($L, T \to \infty$, typically $L \sim T/2 \sim 6\,\text{fm}$ to possibly $11\,\text{fm}$)
  - tuning the QCD scale and the quark masses to their physical value
  - strong-isospin-breaking ($m_u \neq m_d$) and QED corrections $O(\alpha)$



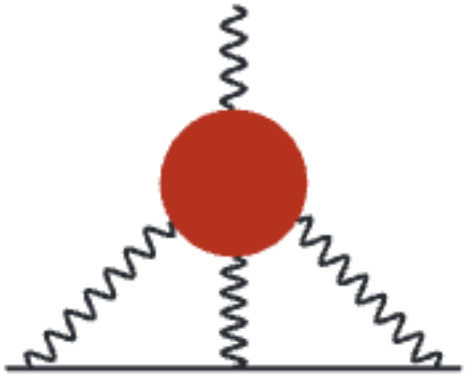
- Use effective field theories (EFTs) to guide extrapolations
  - Symanzik EFT for $a \to 0$: expansion of $\langle O \rangle^{\text{lat}}$ in $a^m \alpha_s(a)^n$, $m \geq 2$, $n \geq 0$
  - Chiral perturbation theory for $L, T \to \infty$ (effects determined by pions, the lightest hadron): expansion in $M_\pi^2/(4\pi F_\pi)^2 \sim 0.01$
  - ...

Hadronic light-by-light contribution: a_μ^{HLbL}





HLbL contribution to a_μ



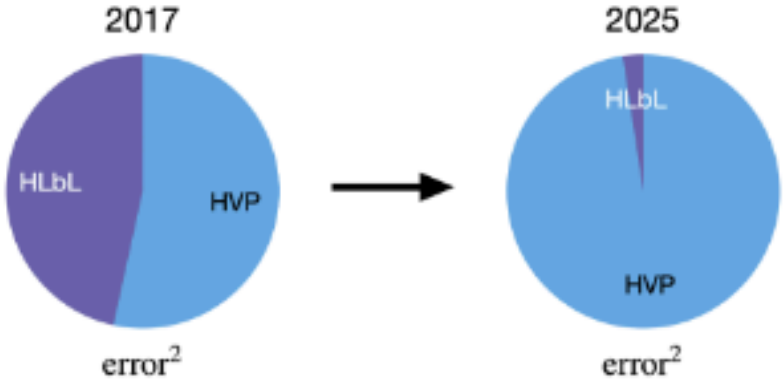
- Enters at $O(\alpha^3)$
  - $\sim$ 70 times smaller than HVP
  - target relative precision $\sim$ 10% instead of $\sim$ 0.2%
- Challenging
  - low-energy hadronic physics $\rightarrow$ nonperturbative methods
  - complex Lorentz structure: $\Pi_{\mu\nu\rho\sigma}(q_1, q_2, q_3) = \int_{xyz} \Pi_{\mu\nu\rho\sigma}(x, y, z) \times e^{-i(q_1 x + q_2 y + q_3 z)} \rightarrow$ 41 form factors (12 contribute to a_μ^{HLbL})
  - complex analytic structure

- Until 2016: model estimates $\rightarrow$ uncontrolled errors

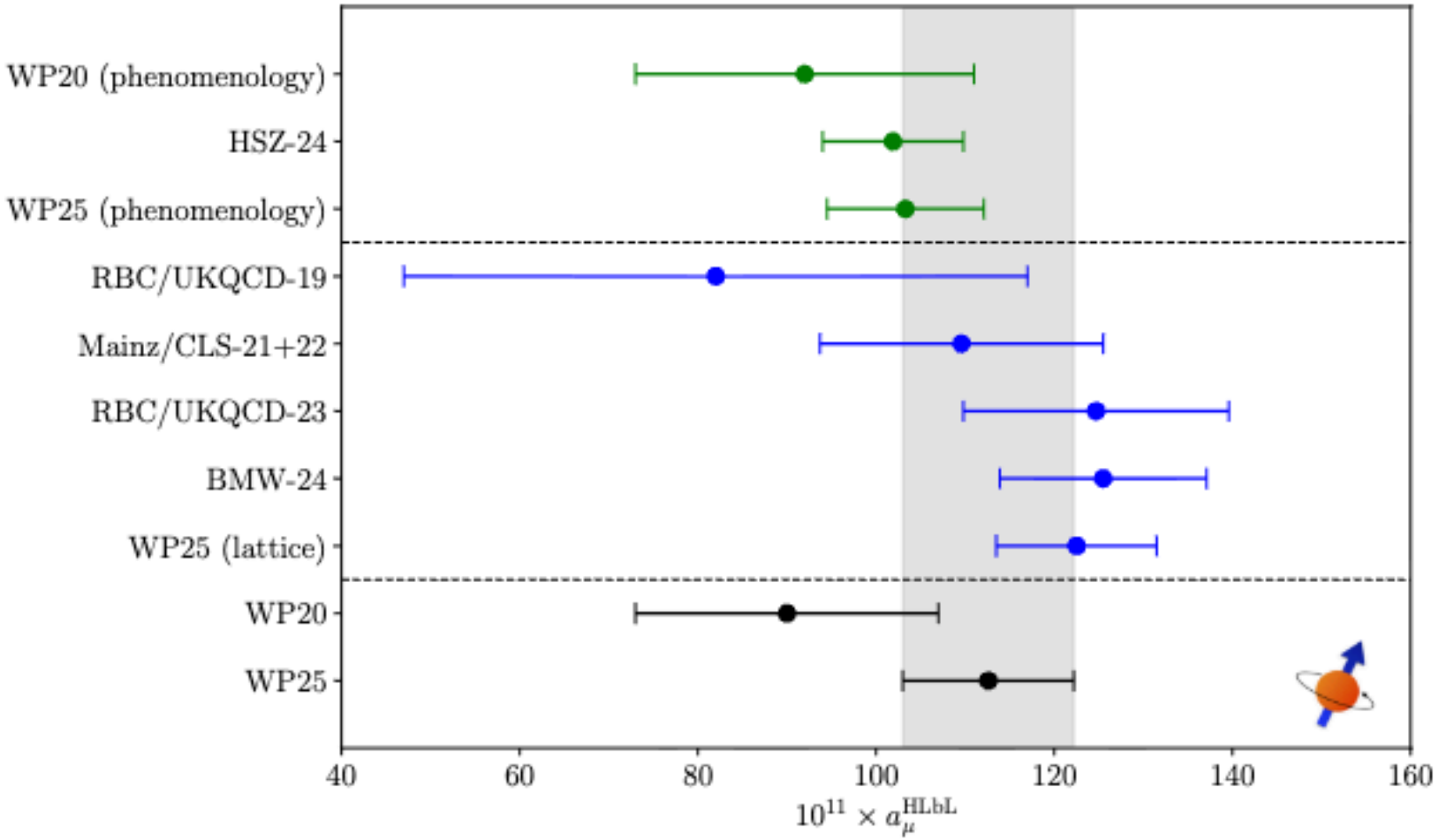
$$a_\mu^{\mathrm{HLbL}} = 105(26) \times 10^{-11} \quad \text{[Prades, de Rafael, Vainshtein '09]}$$

$$a_\mu^{\mathrm{HLbL}} = 116(39) \times 10^{-11} \quad \text{[Jegerlehner \& Nyffeler '09]}$$

- Today: dispersive determinations (mostly model-independent) & lattice QCD calculations
- <10% precision reached in 2025

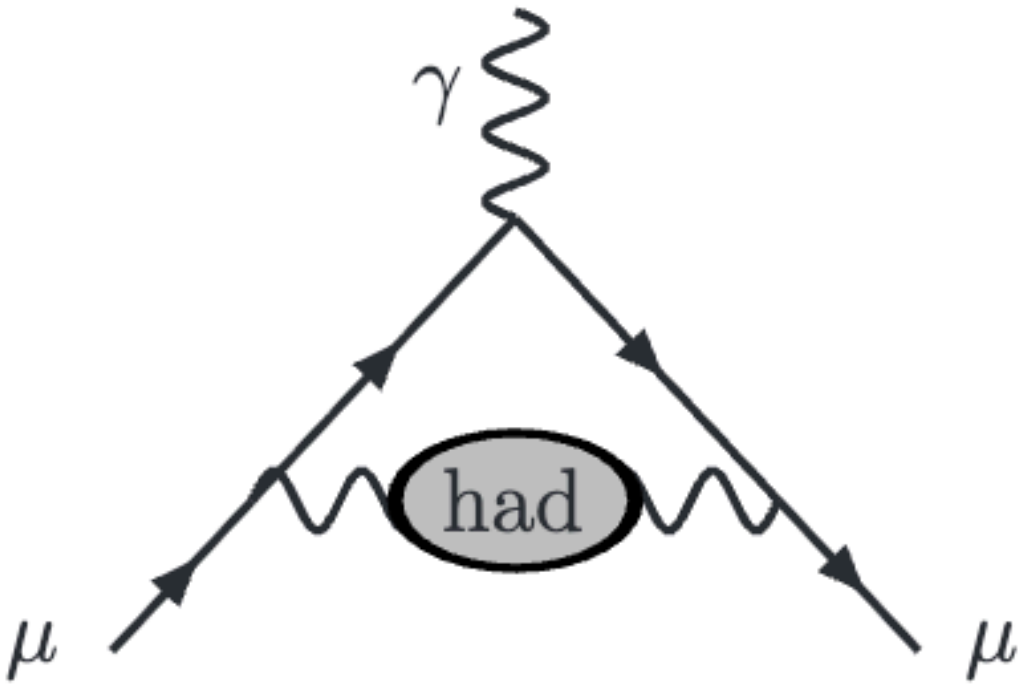


WP 2025 summary of a_μ^{HLbL} contribution



$$a_\mu^{\mathrm{HLbL}} \times 10^{10} = 11.26 \pm 0.96 \quad [8.5\%]$$

Hybrid calculation of $a_\mu^{\text{LO-HVP}}$: making the best of lattice & dispersive approaches



All quantities related to a_μ will be given in units of 10^{-10}

$a_\mu^{\text{LO-HVP}}$ from LQCD: introduction



Compute on $T \times L^3$ Euclidean-spacetime lattice w/ spacing a [Bernecker et al '11]

$$C_L(t) = \frac{a^3}{3} \sum_{i=1}^{3} \sum_{\vec{x}} \langle J_i(x) J_i(0) \rangle$$

w/ $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots$

Decompose

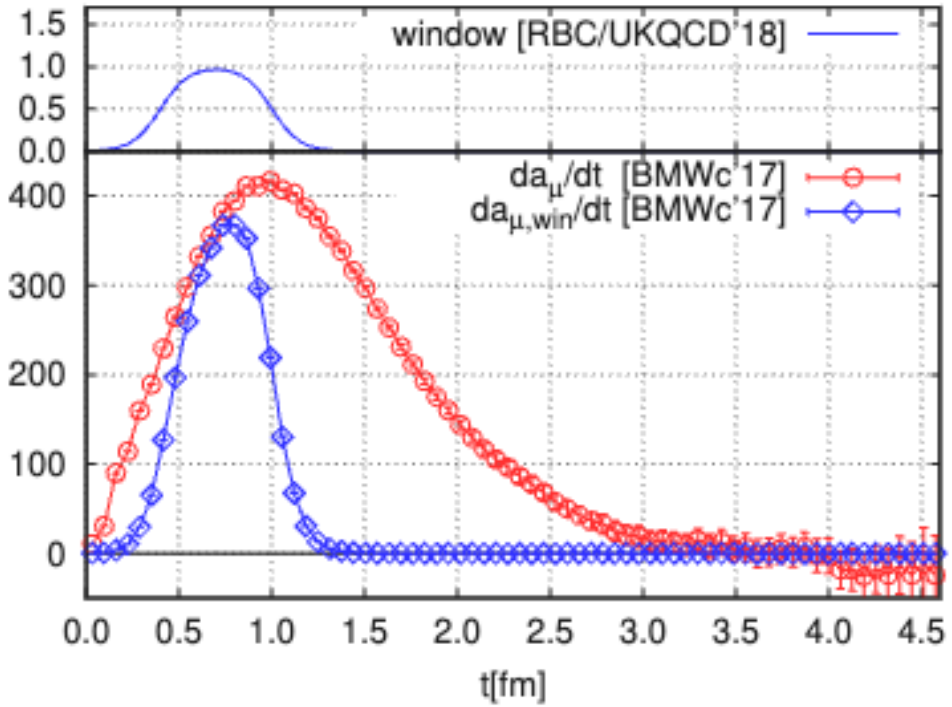
$$C_L(t) = C_L^{ud}(t) + C_L^{s}(t) + C_L^{c}(t) + C_L^{\text{disc}}(t) + C_L^{\text{QED}}(t) + C_L^{\text{SIB}}(t)$$

Then get

$$a_{\mu,f}^{\text{LO-HVP}} = \lim_{\substack{a \to 0 \\ L,\, T \to \infty}} \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{a}{m_\mu^2}\right) \sum_{t=0}^{T/2} K(t m_\mu)\, \text{Re} C_L^f(t)$$

Define "windows" [RBC/UKQCD '18]

$K(\tau) \to W(\tau; \tau_i, \tau_f, \bar{\Delta}) K(\tau)$



Challenges

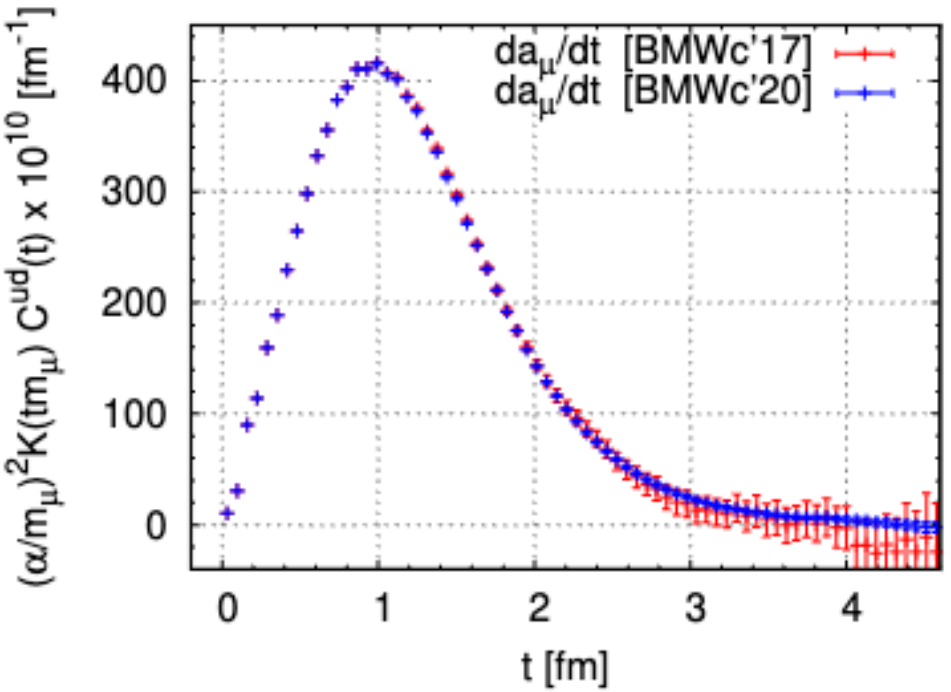
(a) Statistical uncertainties of light and disconnected contributions large t

(b) Finite V (and T) corrections on $I = 1$ contribution

(c) Continuum limits

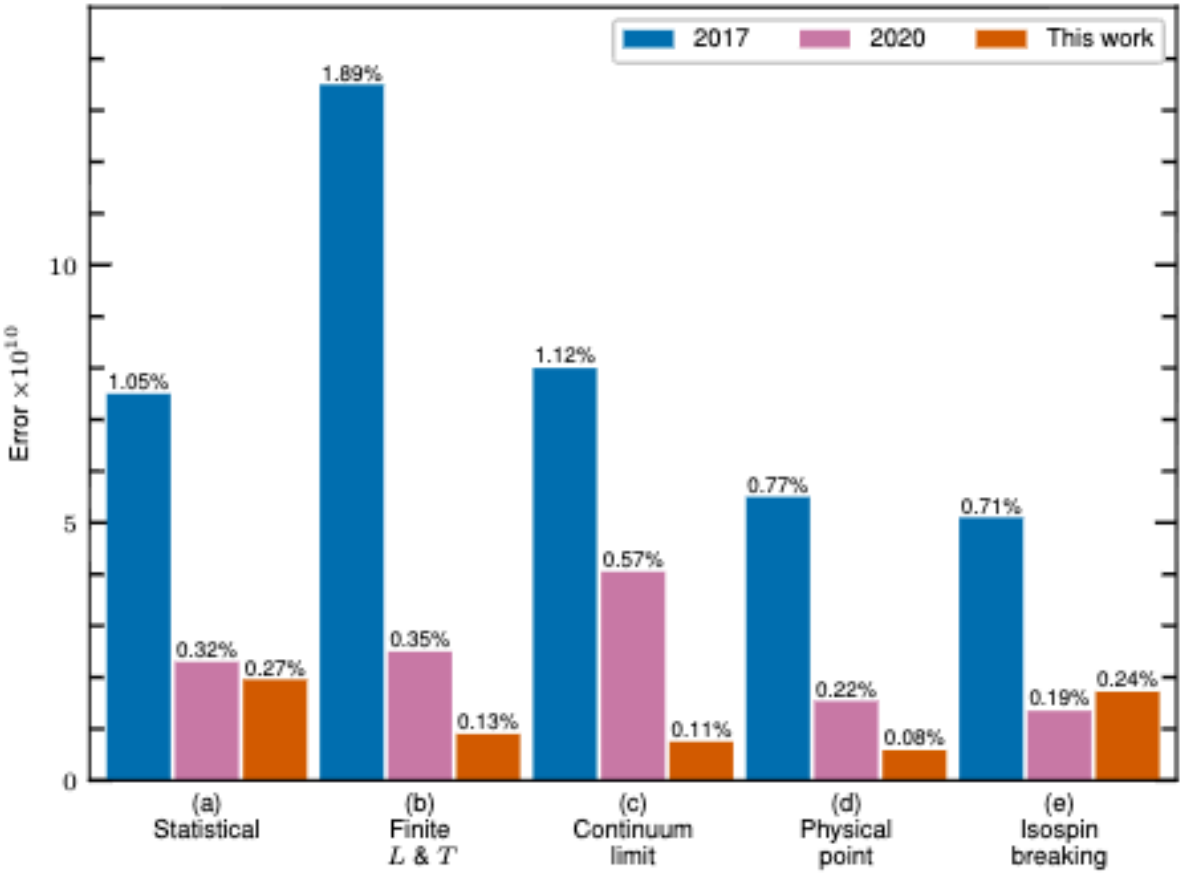
(d) Tuning of physical point $\leftrightarrow$ very precise determination QCD parameters: scale and m_u, m_d, m_s, m_c masses

(e) Must include small effects from electromagnetism (QED) and due to fact that masses of u and d quarks are not quite equal (SIB)



Uncertainty reduction

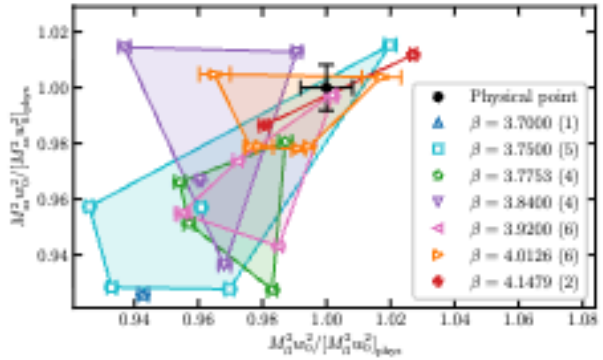
2017 → 2020 → 2024



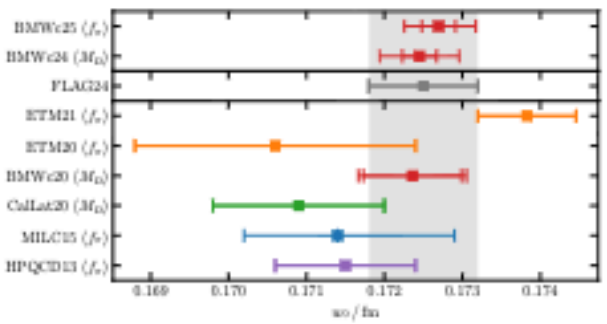
⇒ uncertainty reduced by:

- 2017 → 2020: ÷3.4 or 19. → 5.5
- 2020 → 2024: ÷1.6 or 5.5 → 3.4

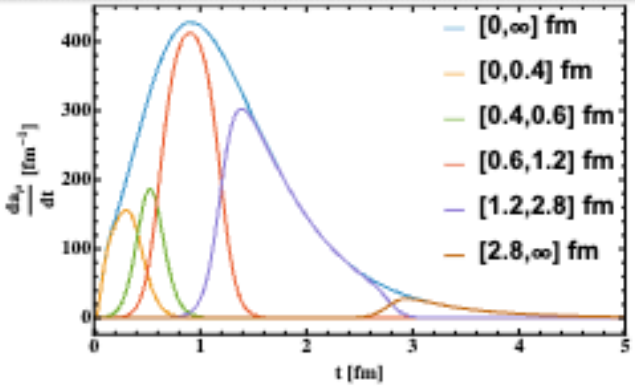
Strategy for improvement over BMW '20



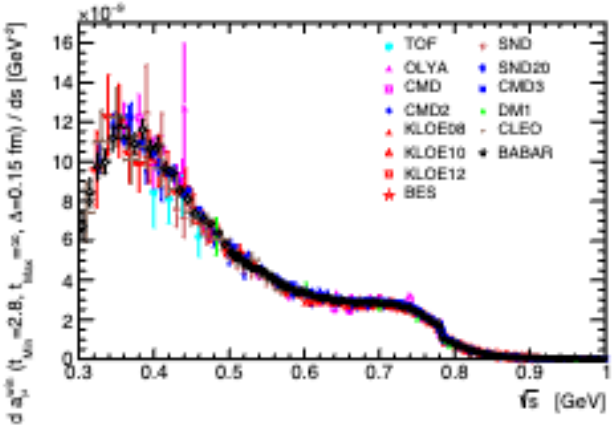
New finer ("Monster") lattices:
$a = 0.064\,\text{fm}$ $[96^3 \times 144] \longrightarrow$
$a = 0.048\,\text{fm}$ $[128^3 \times 192]$
$\rightarrow$ distance to continuum limit $\div 1.8$



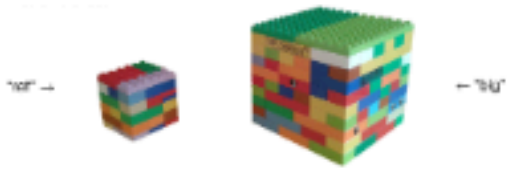
Significantly improve tuning of QCD parameters
$\rightarrow$ (scale uncertainty)^2 $\div 2.1$



Break up calculation into set of 5 windows
$\rightarrow$ optimize calculation for physics relevant in each distance interval



Use $e^+e^- \rightarrow$ hadrons (and τ) data for long-distance 2.8 fm $\rightarrow \infty$ contribution [1fm $\rightarrow \infty$ proposed in RBC/UKQCD '18]



Dedicated simulations on huge $L = 11$ fm lattice to correct finite-V effects w/ many model independent checks



[ericwilliambarnum.wordpress.com]

Calculation fully blinded

Re-evaluated all QED and SIB corrections

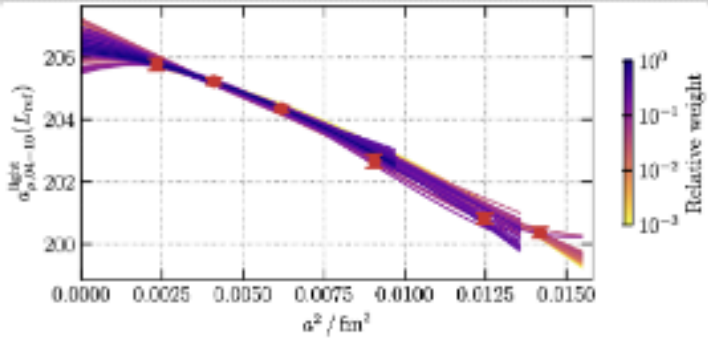
Example analysis: intermediate window benchmark

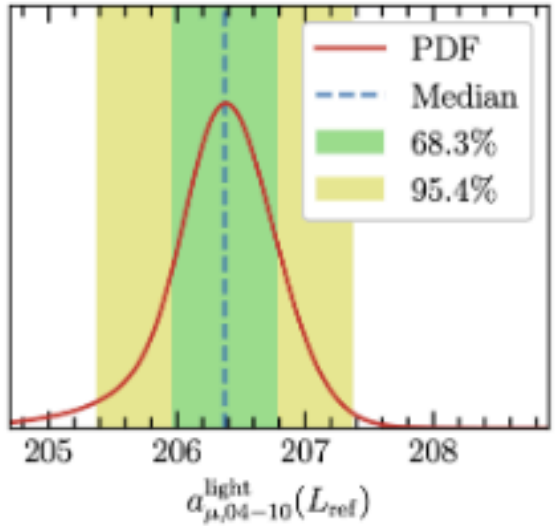
- Euclidean time window $0.4 \to 1.0\,\mathrm{fm}$
  - Contains bulk of experimentally problematic ρ-resonance contribution to $a_\mu^{\text{LO-HVP}}$
  - Cleanest observable on lattice to contain this resonance contribution

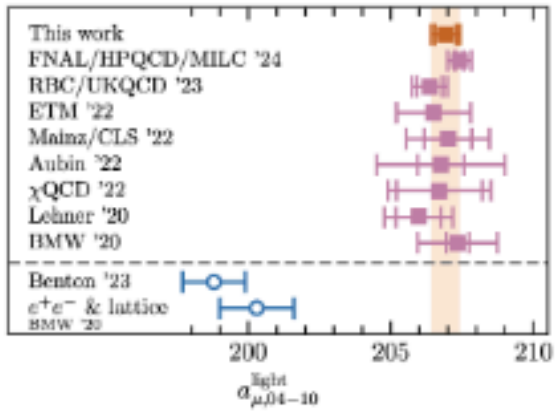
  $\to$ Good benchmark for lattice and dispersive calculations

- Rigorous estimate of uncertainties
  - Statistical uncertainty from jackknife resampling
  - Tens of thousands of analysis variations to account for systematics
  - Each assigned a weight based on Akaike Information Criterion (AIC) $\to$ histograms
  - Consider one/two sigma confidence intervals
  - Total uncertainy is half/quarter of widths (whichever is largest)

- Excellent agreement among lattice calculations (good w/ CMD-3, not shown), but $> 4.5\sigma$ tension with 2020 data-driven analyses





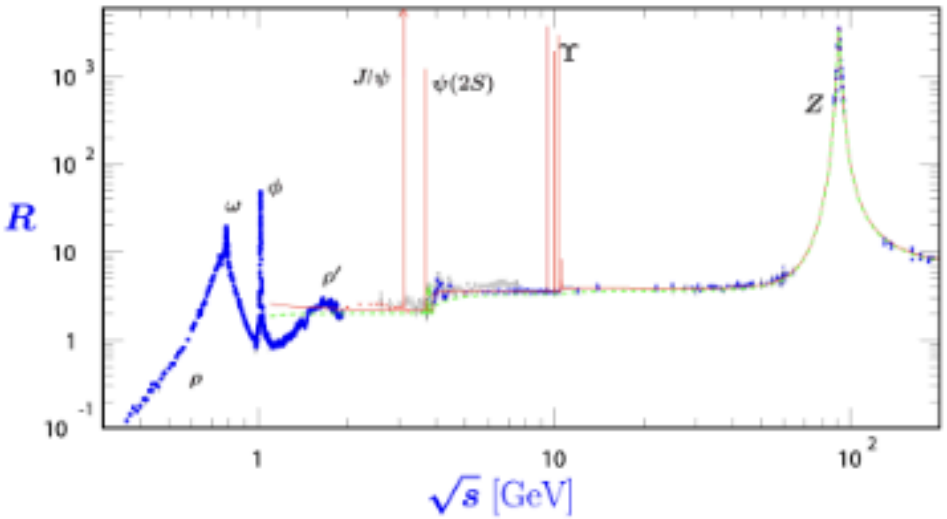


Windows from R-ratio

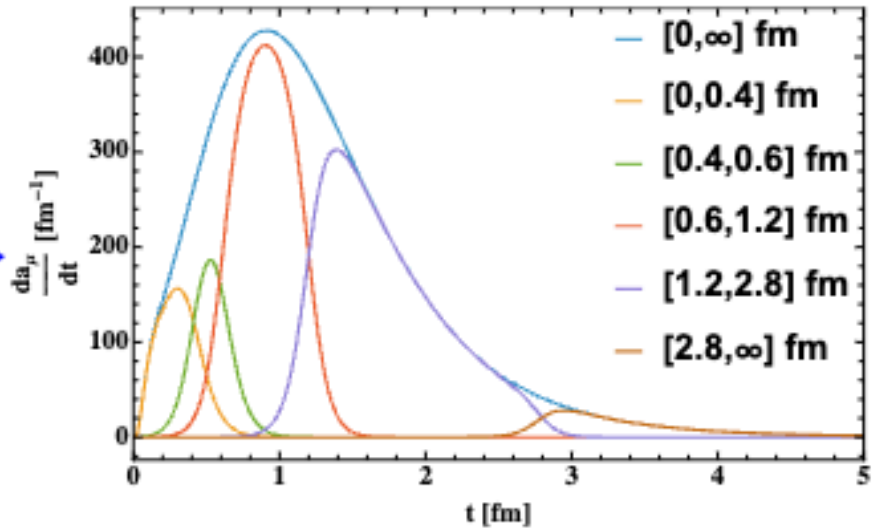
[Berneke '11, RBC '18]

$$a_{\mu,win}^{\text{LO-HVP}}(t_i, t_f) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{dt}{m_\mu^2} K(tm_\mu) W(t; t_i, t_f) C(t)$$

$$R(s) \equiv \frac{\sigma(e^+e^- \to \text{had})}{\sigma(e^+e^- \to \mu^+\mu^-)} \quad \longrightarrow \quad C(t) = \frac{1}{24\pi^2} \int_{s_{\text{th}}}^\infty ds \sqrt{s}\, R(s)\, e^{-|t|\sqrt{s}}$$



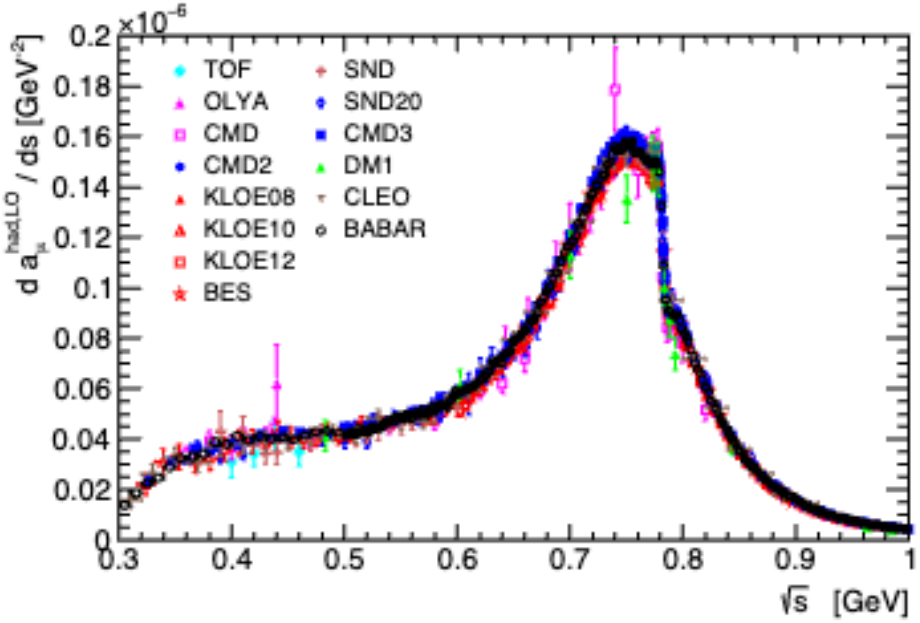
[PDG '18 compilation]



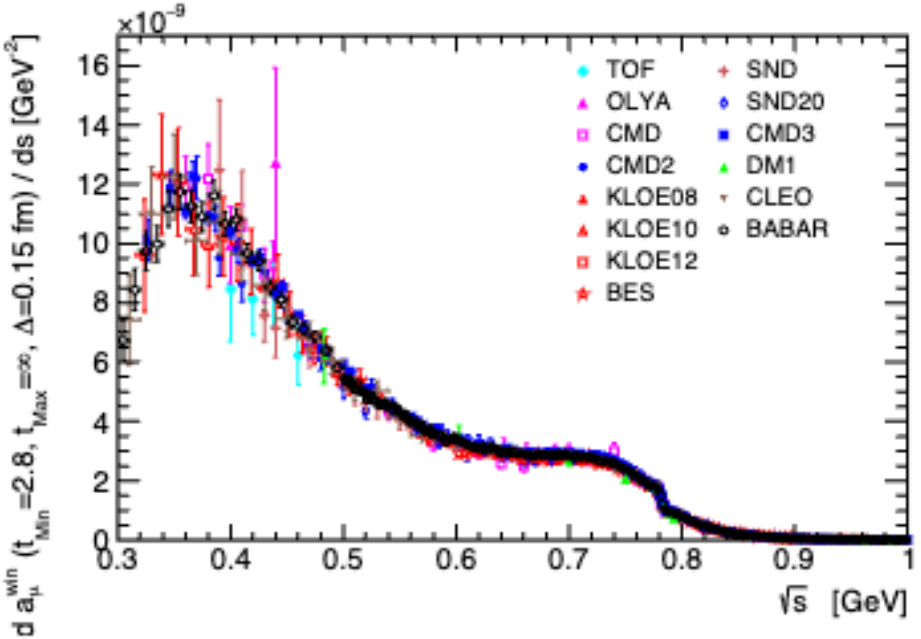
- Compute 4 windows from $0 \to 2.8\,\text{fm}$ on lattice
- Obtain small contribution from $2.8\,\text{fm} \to \infty$ using $e^+e^- \to$ hadrons data

Data-driven tail: 2.8 fm $\rightarrow \infty$

$a_\mu^{\text{LO-HVP}}$



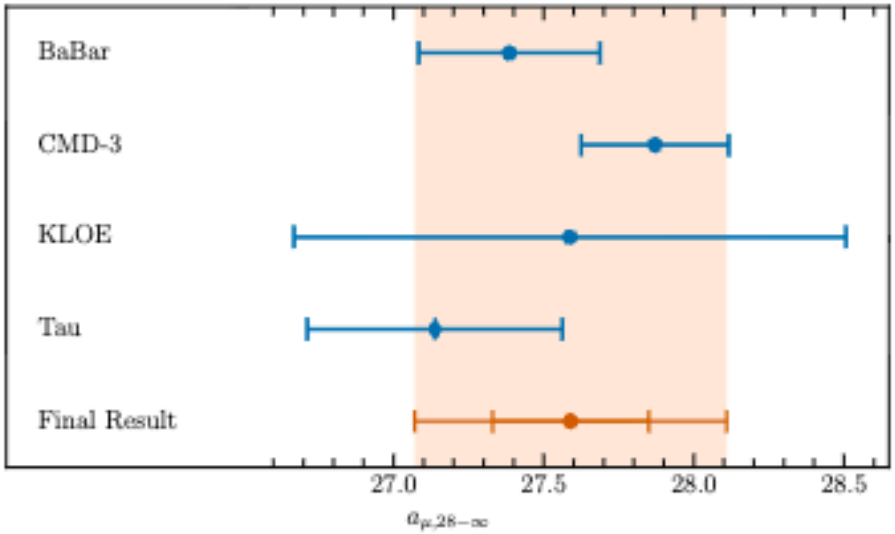
$a_{\mu,28-\infty} \equiv a_\mu^{\text{LO-HVP}}|_{\text{tail}}$



- Over 70% of tail comes from below problematic ρ peak
- All datasets agree well in that range
- Good agreement with lattice for $a_{\mu,28-35}$
- Contributes only 4% to $a_\mu^{\text{LO-HVP}}$
- 96% comes lattice computation
- Helps reduce many of our uncertainties

Data-driven tail: result

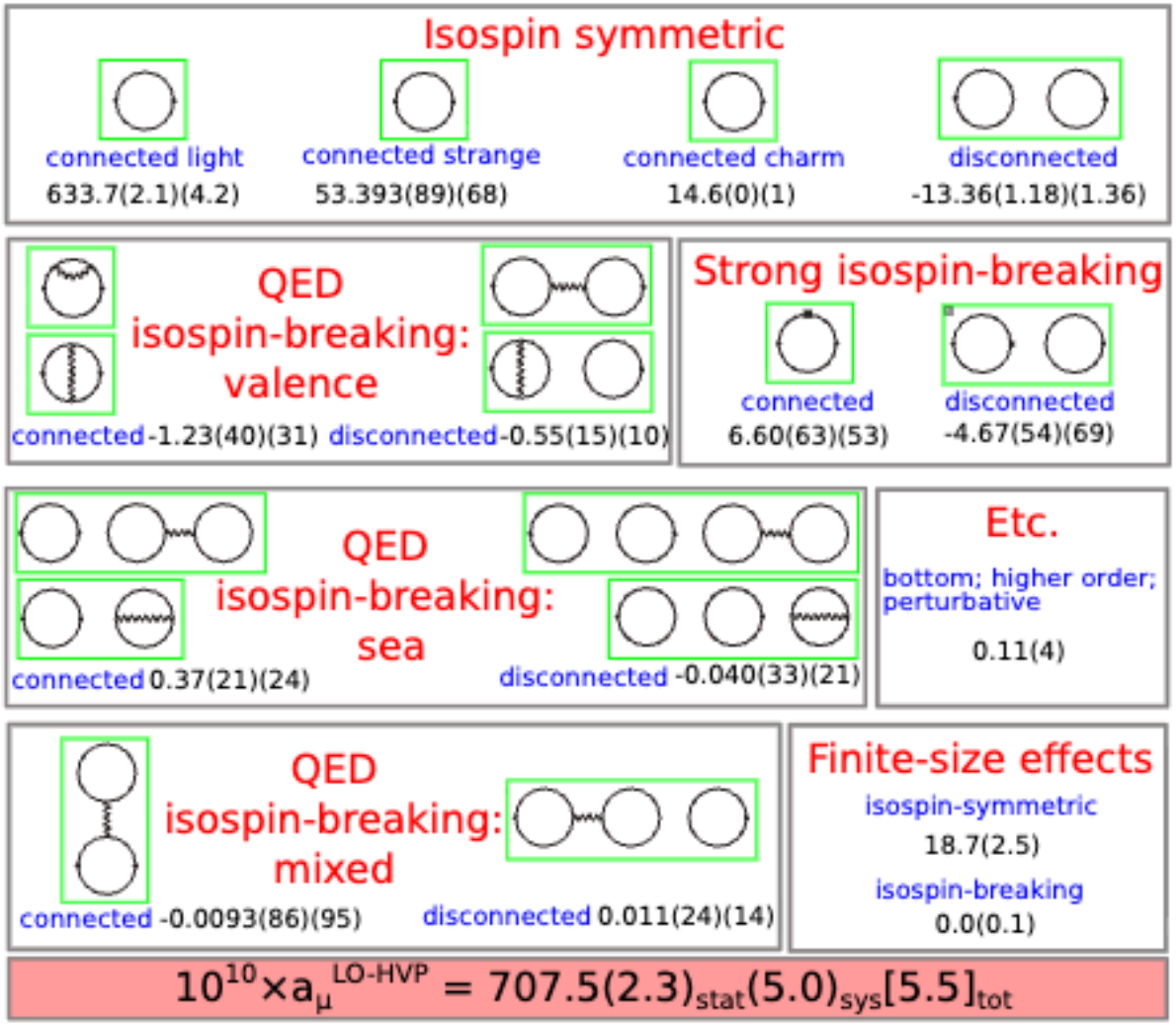
- Central value and first set of errors: only experiments with $\geq 95\%$, $\sqrt{s}$ coverage of the dominant $\pi^+\pi^-$ contribution
- Here, all experiments give compatible results $(\chi^2/\text{dof} = 1.0)$
- Include additional systematic variations
  - Averaging the spectrum before integrating it to give $a_\mu^{\text{LO-HVP}}$ instead of integrating, then averaging
  - Effect of experiments with less coverage
  - Removing/including CMD3
  - Removing BaBar/KLOE
- Still negligible effect on our final result for $a_\mu^{\text{LO-HVP}}$



Integral of avg.	0.12
Other exp.	0.16
CMD3	0.31
BaBar/KLOE	0.26

$$a_{\mu,2.8\text{-}\infty\,\text{fm}}^{\text{LO-HVP}}|_{e^+e^-,\tau} = 27.59(26)(45)[52]$$

Summary of contributions to $a_\mu^{\text{LO-HVP}}$: 2020



$a_\mu^{\text{LO-HVP}}$ to 0.77% in 2020

[Frontiers of Science Award in Theoretical Physics (2024)]

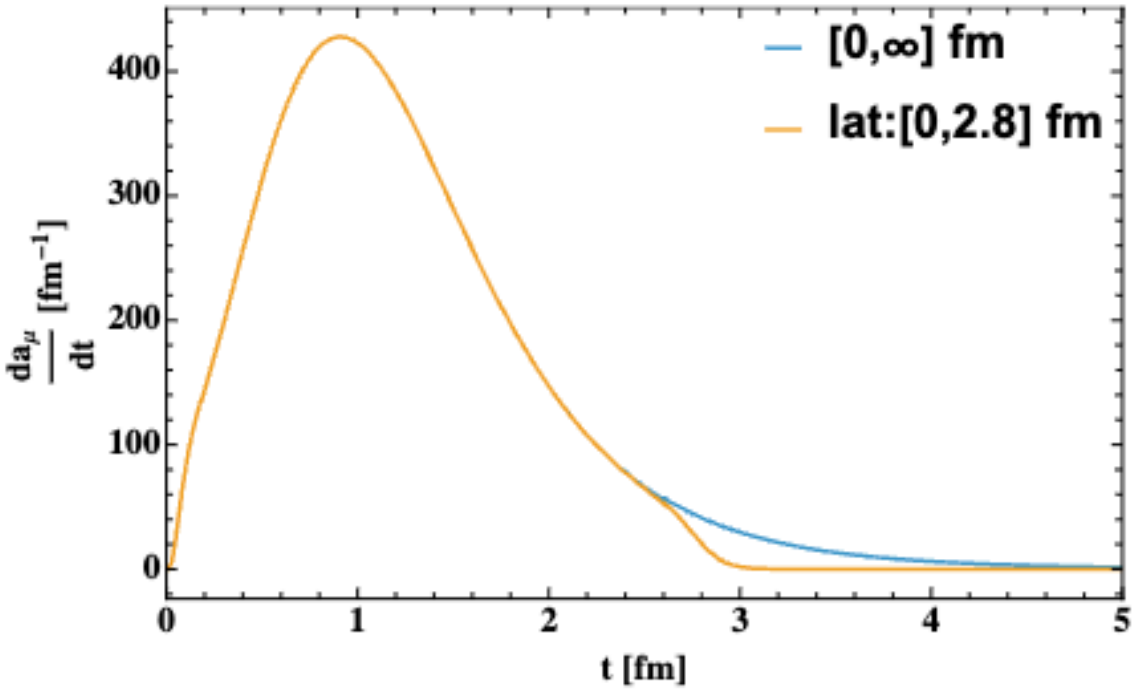
July 12, 2024: unblinding



Preprint uploaded to arXiv on July 15, 2024

LO-HVP contribution result

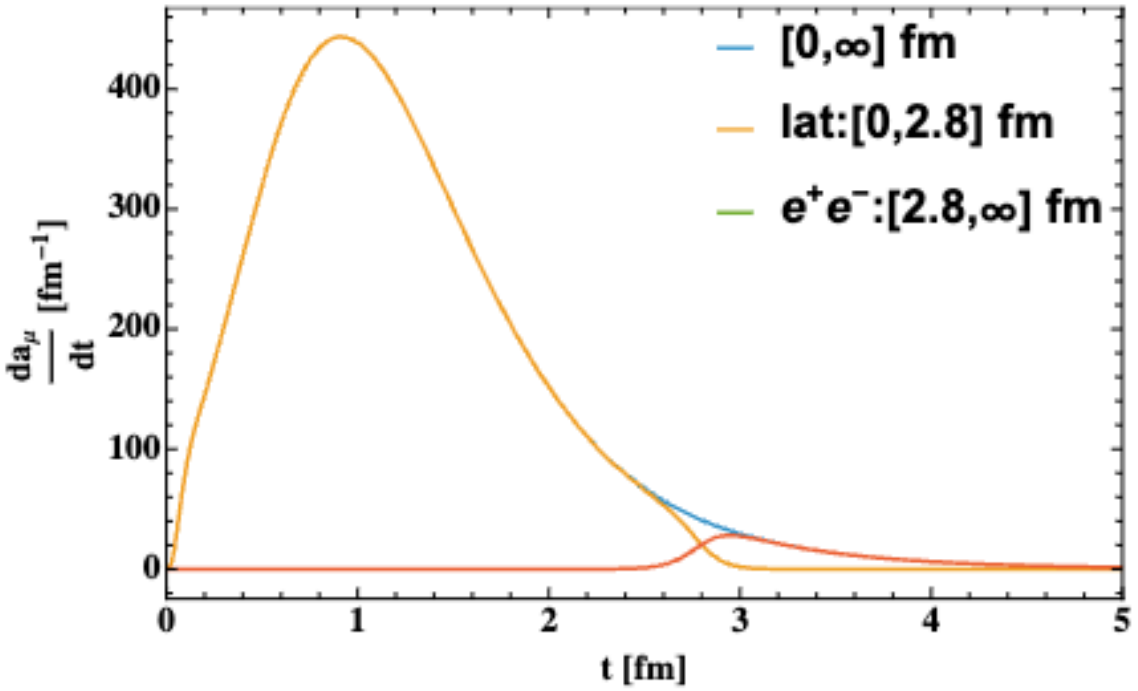
Re-computed all lattice LO-HVP contributions [BMW-DMZ '24]



$$a_\mu^{\text{LO-HVP}} \equiv a_{\mu,\text{0-2.8 fm}}^{\text{LO-HVP}}\big|_{\text{lat}}$$
$$= 687.51(3.36)$$
$$= 96\%$$

LO-HVP contribution result

Re-computed all lattice LO-HVP contributions [BMW-DMZ '24]



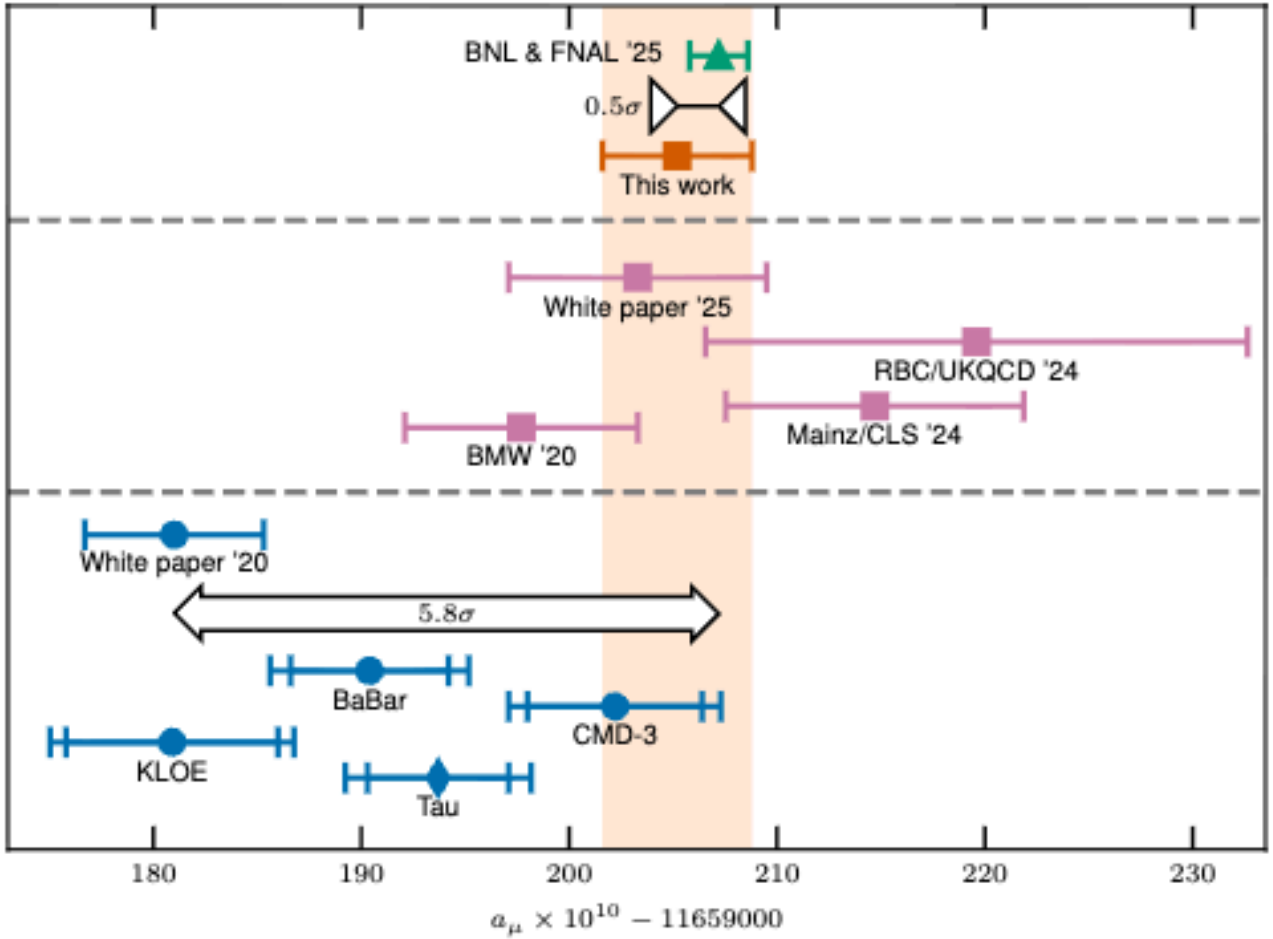
$$
\begin{aligned}
a_\mu^{\text{LO-HVP}} &\equiv a_{\mu,\text{0-2.8 fm}}^{\text{LO-HVP}}|_{\text{lat}} + a_{\mu,\text{2.8-}\infty\text{ fm}}^{\text{LO-HVP}}|_{e^+e^-,\tau} \\
&= 687.51(3.36) + 27.59(52) \\
&= 96\% + 4\% \\
&= 715.1(3.4) \quad [0.48\%]
\end{aligned}
$$

Experiment vs SM: November 2026

$$a_\mu|_{\text{expt}} = (11659207.15 \pm 1.45) \times 10^{-10}$$

$$a_\mu|_{\text{WP25}} = (11659203.3 \pm 6.2) \times 10^{-10} \qquad [\leftarrow a_\mu^{\text{LO-HVP}} \text{ from lattice only}]$$

$$a_\mu|_{\text{This work+WP25}} = (11659205.1 \pm 3.4) \times 10^{-10}$$



Conclusion and outlook

- New "hybrid" calculation (lattice + $e^+e^- \to$ hadrons) of $a_\mu^{\text{LO-HVP}}$ to 0.48%
- a_μ^{HLbL} now known to $< 9\%$ (target reached)
- Decades of perturbative QED & EW calculations, combined w/ fully nonperturbative QCD (+QED) and data-driven ones, predict g_μ to 0.36 ppb !
- Decades of measurement give g_μ to 0.14 ppb ! $\to$ 2026 Breakthrough Prize
- SM confirmed to 0.38 ppb
- Most stringent, single test of the complete SM (all particles & interactions)

Conclusion and outlook

- Still need $\times 2$ improvement on $a_\mu^{\text{LO-HVP}}$ to match experiment !
- Significant work needed to understand tensions among e^+e^- measurements on ρ peak [BaBar '23, Colangelo et al, ...] and w/ lattice [ETM '22, BMW-DMZ '23, ...]
- Eagerly await
  - Next generation of 100% lattice calculations w/ precision $\sim$BMW-DMZ '24
  - New BABAR $e^+e^- \to$ hadrons analysis
  - New KLOE analysis
  - New BES III, BELLE-II, CMD-3, SND-2 data and analysis
  - J-PARC entirely new method for a_μ measurement
  - MUonE @ CERN for spacelike HVP
- $\alpha(M_Z^2)$ for EWPO tests of SM @ FCC-ee
- ...