Hadronic vacuum polarization to three loops in chiral perturbation theory

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JHEP 02 (2026) 154 & JHEP 08 (2026) 116



Introduction and motivation

Hadronic vacuum polarization (HVP)

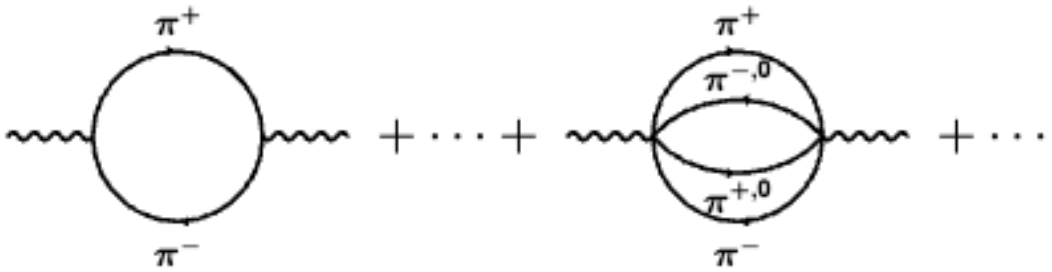
$$\Pi_{\mu\nu}(q) \;=$$



$$= \int d^4x\, e^{iq\cdot x} \langle 0|T\{J_\mu(x)J_\nu(0)\}|0\rangle$$

$$= \left(q_\mu q_\nu - \eta_{\mu\nu} q^2\right) \Pi_T(q^2)$$

$\xrightarrow{q^2\to 0,\ m_u=m_d,\ \alpha=0}$



w/ $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \cdots$

2π (& 4π) dominate at small $-2M_K^2 \lesssim q^2 \lesssim 2M_K^2$ ...

... e.g. at long Euclidian distances which are particularly noisy on lattice...

... but heavenly for SU(2) χPT

Low-q^2 HVP relevant:

$\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$:

- hadronic contribution to $\alpha(0) = \alpha \longrightarrow \alpha(M_Z^2)$
- needed to $\sim 0.1\%$ (or better) for FCC-ee

$a^{\rm HVP}_{\mu,e}$:

- hadronic contribution to $a_{\mu,e} = (g_{\mu,e} - 2)/2$
- needed to $\sim 0.2\%$ to match experimental precision on a_μ

In both cases, uncertainties dominated by nonperturbative low-energy QCD

$\Rightarrow$ need all available help

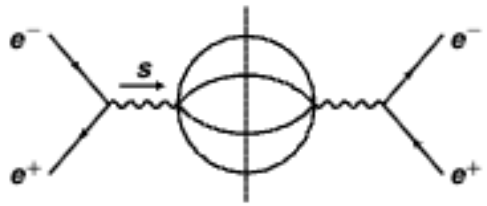
Example applications

N^{3}LO: first order that gives FV correction to a_μ in $L_{\text{ref}} = 6.272\,\text{fm}$ w/ uncertainty $< 1.45 \times 10^{-10}$, i.e. a_μ measurement error

- know FV corrections at NLO & NNLO in SU(2) χPT
- NLO: $(12.4 \pm 4.3) \times 10^{-10}$
- NNLO: $(16.7 \pm ??) \times 10^{-10}$
- assuming geometric progression of series: N^{3}LO $\to (18.2 \pm 0.5) \times 10^{-10}$

[2 LEC combinations: l_6 (NLO, known); $2c^r_{50} - c^r_{56}$ (2-loop VV & AA ffs) (NNLO)]

N^{3}LO: first order that has 4-pion cut



$$\sim \text{Im}\Pi(s)$$

$$\sim \frac{d\sigma}{ds}(e^+e^- \to 4\pi)$$

[No LECs; only M_π & F_π]

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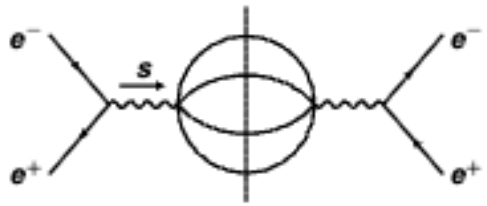

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The challenge

PHYSICAL REVIEW D **102,** 094511 (2020)

Application of effective field theory to finite-volume effects in a_μ^{HVP}

Christopher Aubin,[1] Thomas Blum,[2] Maarten Golterman,[3] and Santiago Peris[4]

appear already at two loops, which is the minimum number. While it is unlikely that the full N^{3}LO analysis will ever be worked out in practice, it is important to establish the validity of the EFT framework for the study of a_μ^{HVP} in order to be assured that even the NNLO analysis of finite-volume effects carried out in Ref. [10] has a solid EFT basis. We believe that our discussion in this paper illustrates why indeed this is the case. We expect the same separation

Arthur C. Clarke's 1st law

"*When a distinguished [. . .] scientist [. . .] states that something is impossible, [s]he is very probably wrong.*"

Only prior N^{3}LO calculation is for "much simpler" M_π & F_π [Bijnens et al '17]

χPT setup: lagrangian and HVP loops

χPT lagrangian and power counting

- Two flavors (pions only), non-dynamical photon field
- Vertices with $2n$ pions and up to 2 photons, drawn from

$$\mathcal{L}_{\text{ChPT}} = \mathcal{L}_{\text{LO}} + \mathcal{L}_{\text{NLO}} + \mathcal{L}_{\text{NNLO}} + \mathcal{L}_{\text{N}^3\text{LO}} + \cdots$$

- $\mathcal{L}_{\text{N}^3\text{LO}}$: 475 $\mathcal{O}(p^8)$ counterterms [Bijnens, Hermansson-Truedsson & Wang '19]
- $\mathcal{L}_{\text{NNLO}}$: 57 $\mathcal{O}(p^6)$ counterterms [Bijnens, Colangelo & Ecker '99, '00]
- $\mathcal{L}_{\text{NLO}}$: 7 $\mathcal{O}(p^4)$ counterterms [Gasser & Leutwyler '84]
- $\mathcal{L}_{\text{LO}}$: Basic $\mathcal{O}(p^2)$ vertex

Power counting (the simple way)

N^kLO diagram: $\quad k = (\#\text{ loops}) + \sum_{\text{vertices}} (\#\text{ N's})$

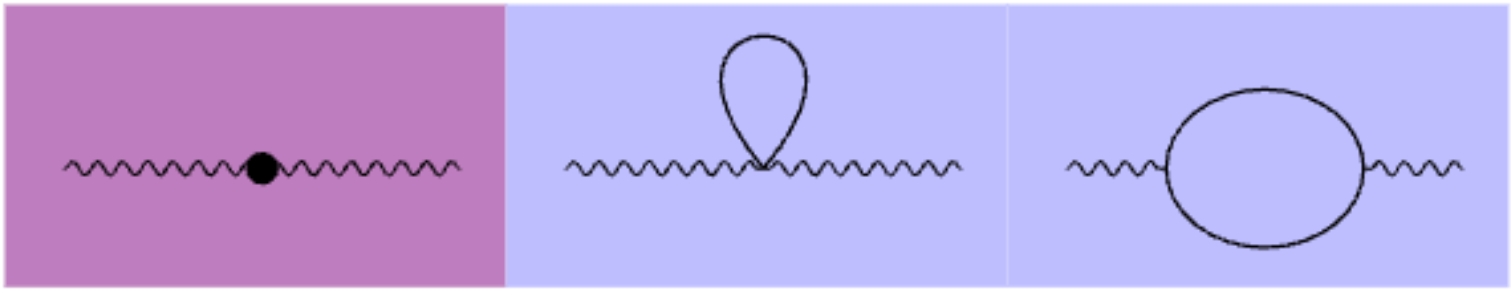
Result organized as

$$\bar{\Pi}_T(t) := \frac{\Pi_T(q^2) - \Pi_T(0)}{\pi_{16}} = \bar{\Pi}_T^{\text{NLO}}(t) + \xi\, \bar{\Pi}_T^{\text{NNLO}}(t) + \xi^2\, \bar{\Pi}_T^{\text{N}^3\text{LO}}(t) + O(\xi^3)$$

$$t := q^2/M_\pi^2, \quad \xi := \pi_{16} M_\pi^2/F_\pi^2 \approx 0.014, \quad \pi_{16} := 1/(16\pi^2)$$

NLO diagrams

3 diagrams & no LEC for $\bar{\Pi}(t)$

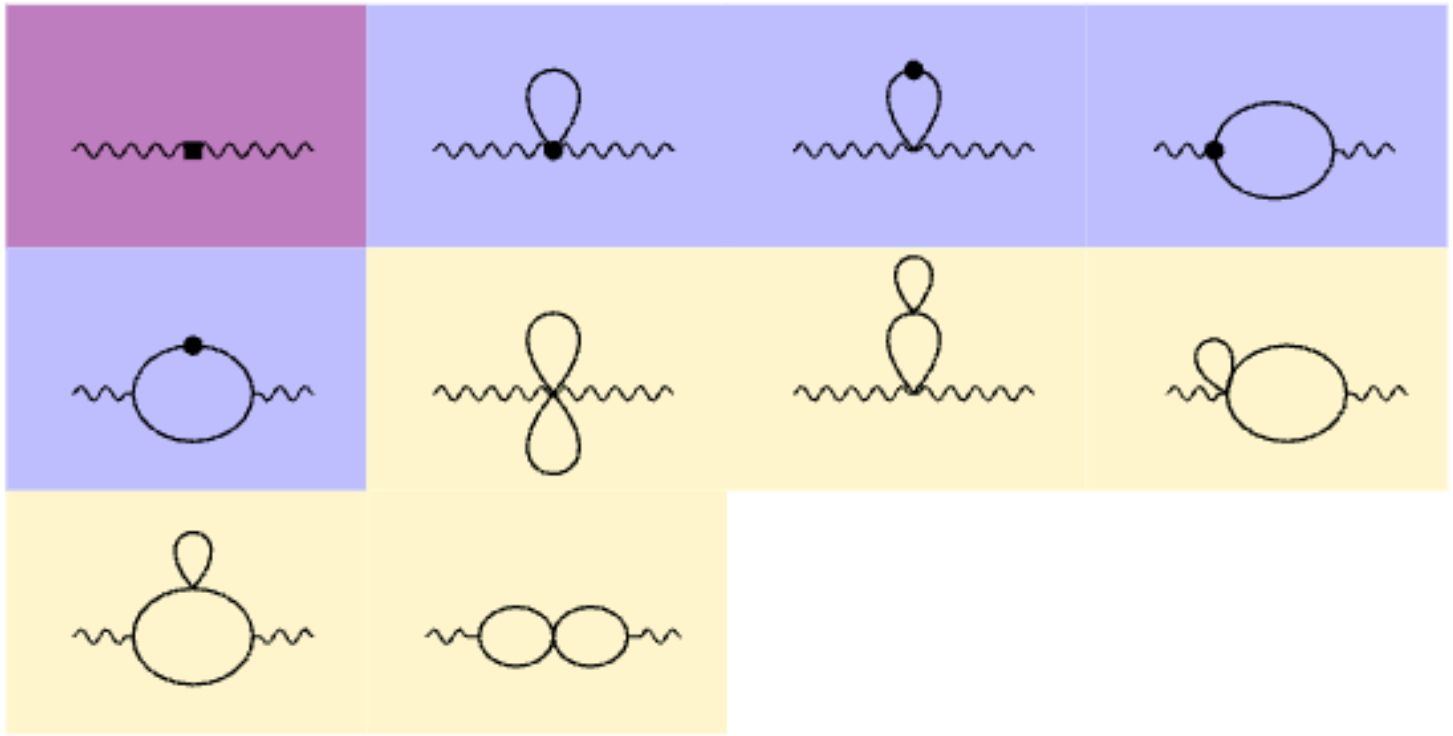


● NLO vertex (unmarked) LO vertex

- : tree, one NLO vertex : one loop, LO vertices
- one-loop integrals: tadpole and bubble
- only the bubble depends on $t = q^2/M_\pi^2$

NNLO diagrams

12 diagrams (mirror images not shown) & 2 LEC combinations for $\bar{\Pi}(t)$



■ NNLO vertex ● NLO vertex (unmarked) LO vertex

: tree, NNLO vertex : one loop, one NLO vertex : two loops, LO vertices

- No genuine two-loop diagrams

NNLO diagrams

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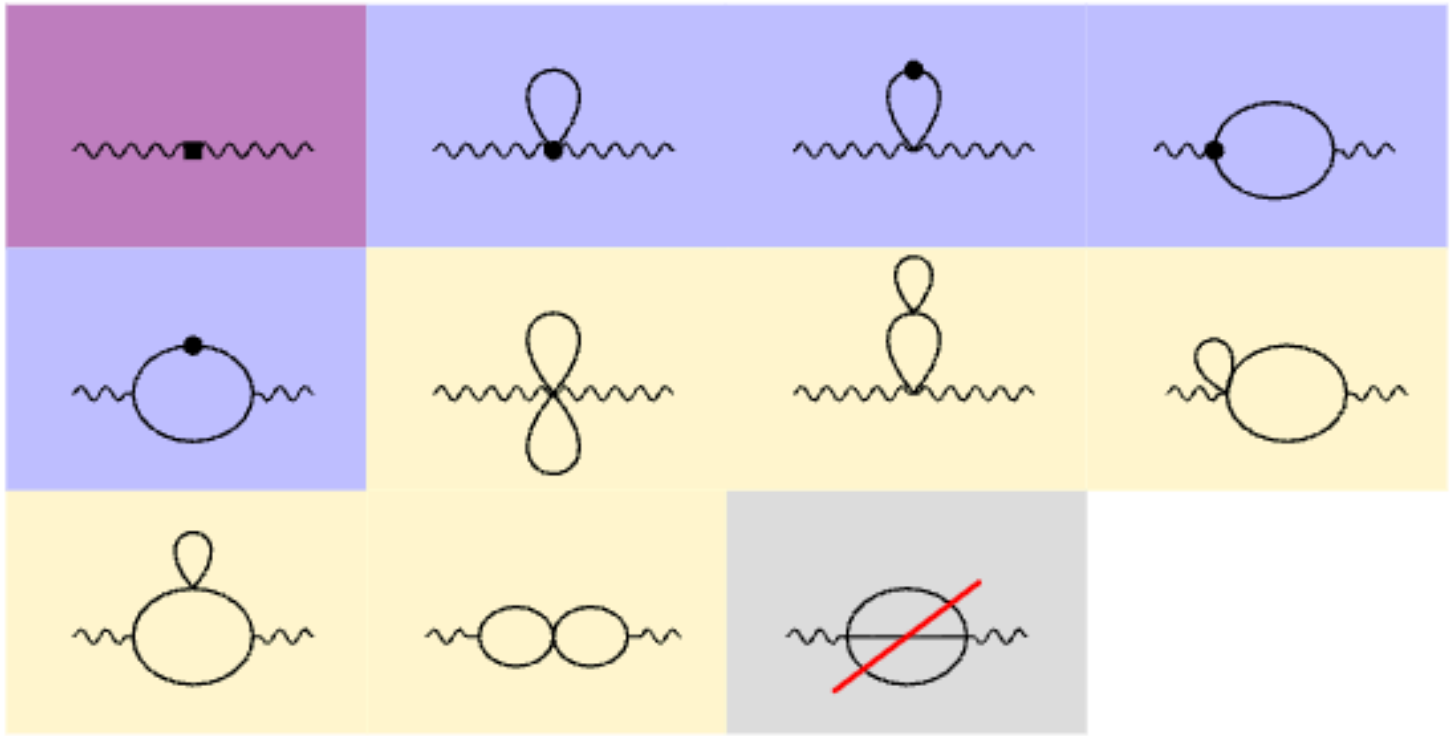


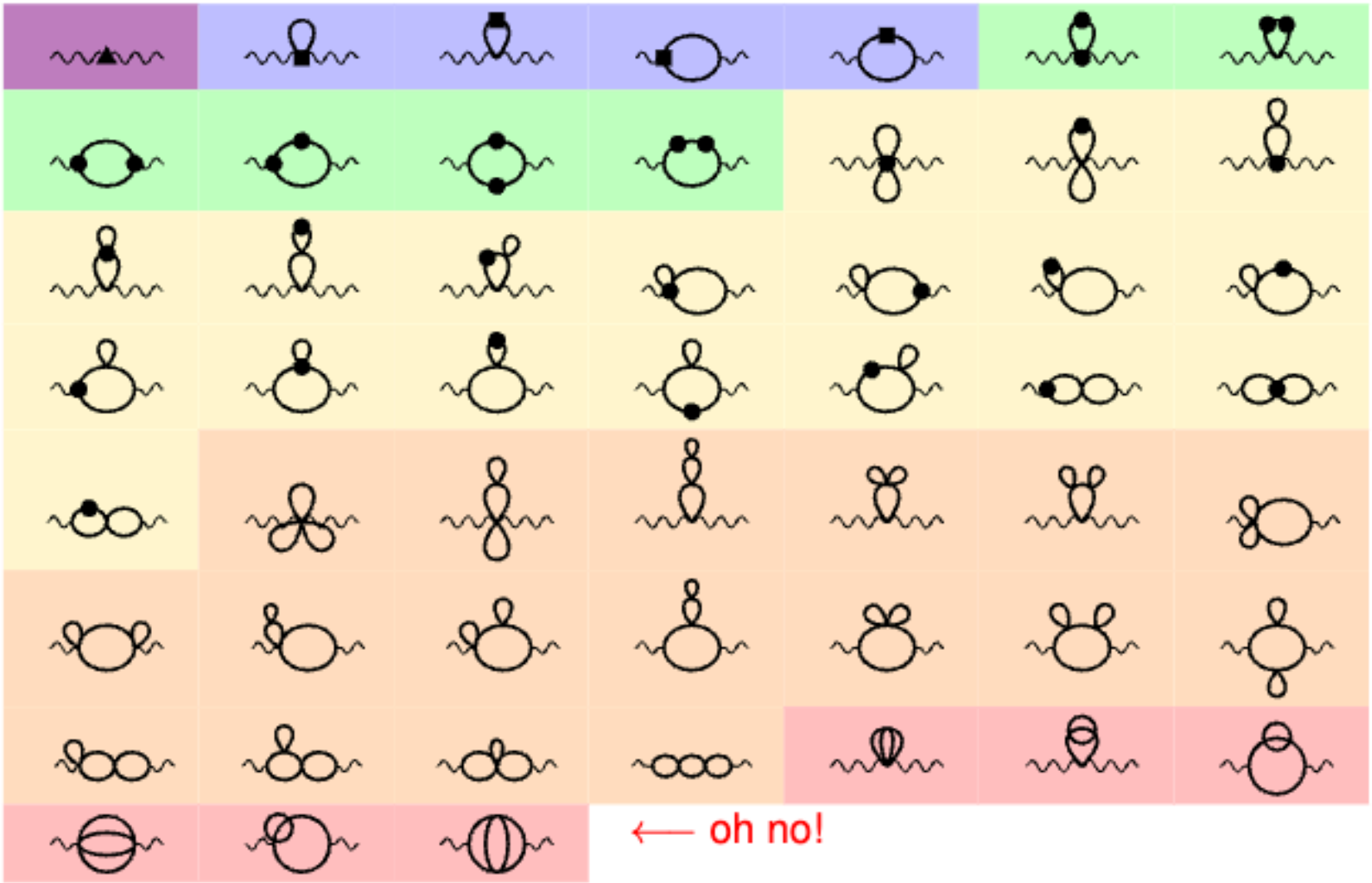
■ NNLO vertex ● NLO vertex (unmarked) LO vertex

: tree, NNLO vertex : one loop, one NLO vertex : two loops, LO vertices

- No genuine two-loop diagrams
- No two-loop sunset: G-parity
  $\Rightarrow$ no vertices with an odd number of pion lines

N^3LO diagrams

102 diagrams (mirror images not shown) & 9 LEC combinations for $\bar{\Pi}(t)$



tree, N^3LO vertex ▲ | one loop, one NNLO vertex ■ | one loop, two NLO vertices ●

two loops, one NLO vertex | three loops, factorizable | three loops, genuine

Loop wrangling

Cool loopology

1. Integration-by-parts (IBP) reduction

All integrals $\to$ finite basis of *master integrals* [Laporta '00; LiteRed 2: Lee '12, '14]

2. Tarasov's dimensional shift

Divergent masters in $d = 4 - 2\epsilon \to$ masters in $d = 2 - 2\epsilon$, finite for the E_i [Tarasov '96]

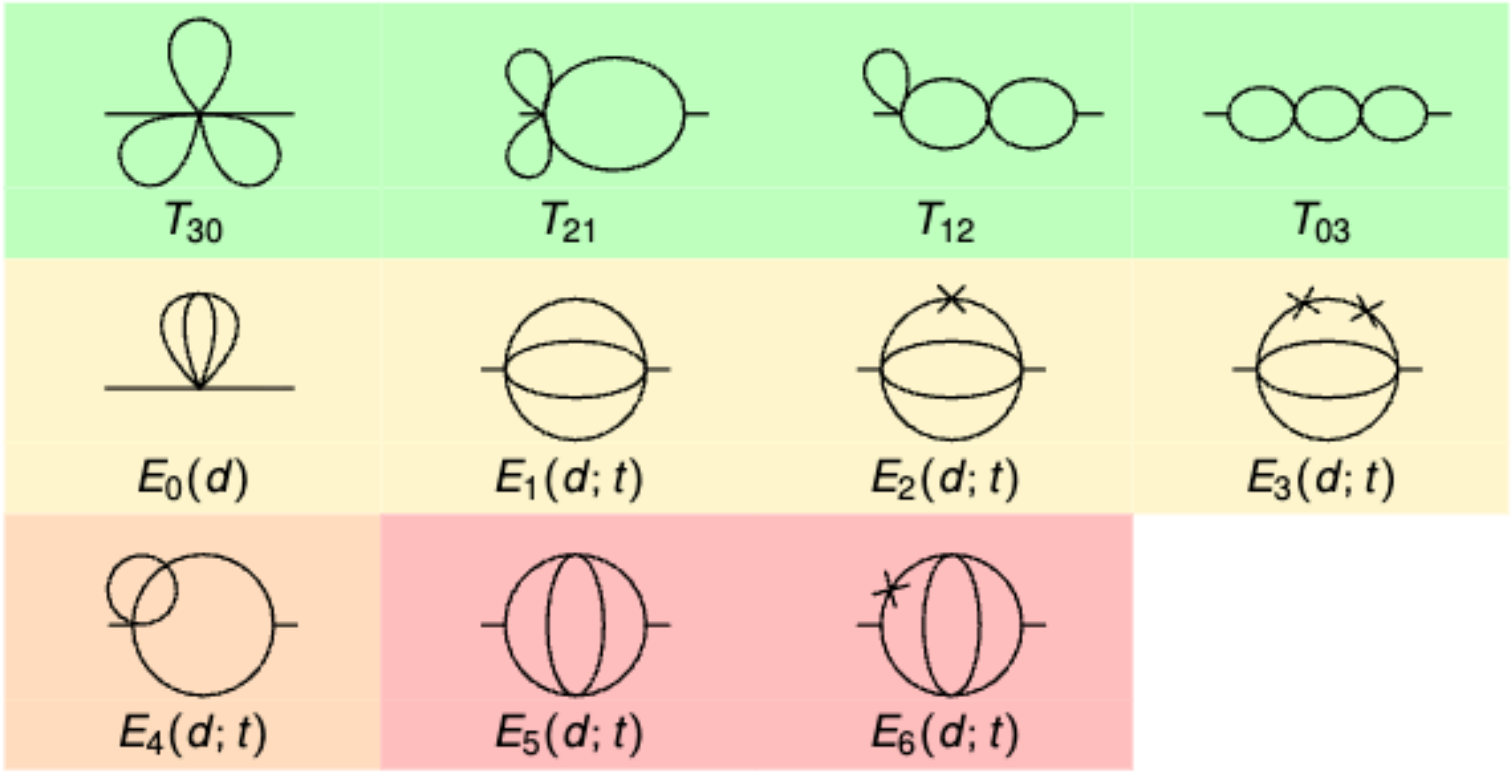
3. Schouten relations

Fixed-d relations not implied by IBP; needed for renormalizability [cf. Remiddi & Tancredi '14]

4. Differential equations

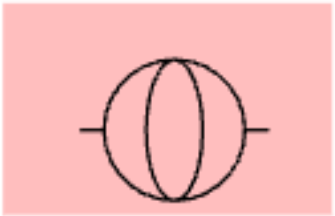
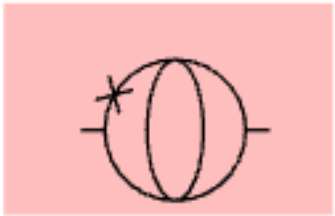
Derivatives in t of masters, IBP-reduced: ODEs in t replace explicit loop integration

Resulting master integrals



- 11 three-loop masters: 4 products $T_{ab} := I^a_{\circ} \, I^b_{\circ}(t)$ (only logarithms & polylogarithms of t), 7 non-factorizable $E_0, \ldots, E_6$ ("E" = elliptic); $\times$ = raised propagator power
- three-loop sunset family E_4 cat's eye
- Linchpin: E_1, the equal-mass three-loop sunset (*banana graph*); all elliptic masters follow from $E_1(2;t)$ by differentiation or integration
- E_4 drops out of the final result; hardest: the cat's eye E_5, E_6

Masters: the cat's eye

 $= E_5(d;t)$  $= E_6(d;t) \sim \partial_t E_5$

- Hardest: $\bar{E}_5(4;t)$ not a combination of $E_n(2;t)$; from its ODE, as one-dimensional integral over $E_1(2;t)$
- Dimensional shift seemingly breaks renormalizability:

$$M_\pi^{-6} E_1(4-2\epsilon;t) = \underbrace{\frac{2}{\epsilon^3} + \frac{23-t}{3\epsilon^2} + \frac{[\text{pol. in } t]}{\epsilon}}_{\text{polynomials in } t\text{: counterterms}} + \overbrace{\bar{E}_1(4;t)}^{\ni E_{1,2,3}(2;t)}$$

$$M_\pi^{-6} E_5(4-2\epsilon;t) = \frac{1}{3\epsilon^3} + \underbrace{\frac{[J^{(n)}, E_n(2;t)]}{\epsilon^2} + \frac{[J^{(n)}, E_n(2;t)]}{\epsilon}}_{\text{elliptic: no counterterm!}} + \bar{E}_5(4;t)$$

How are these divergences cancelled?

Schouten relations

Gram determinant – Schouten identity

$$\mathbb{G}(u,v,w) := \begin{pmatrix} u^2 & u\cdot v & u\cdot w \\ v\cdot u & v^2 & v\cdot w \\ w\cdot u & w\cdot v & w^2 \end{pmatrix} \quad \Rightarrow \quad |\mathbb{G}(u,v,w)| = 0 \ \text{ in } d=2$$

- $|\mathbb{G}(\ell_1,\ell_2,\ell_3)|$ as numerator of an integral finite in $d=2$:

$$G(d;t) := \int \frac{d^d\ell_1\, d^d\ell_2\, d^d\ell_3}{\pi^{3d/2}} \frac{|\mathbb{G}(\ell_1,\ell_2,\ell_3)|}{D_1 D_2 D_4 D_5 D_7^2 D_8}$$

  IBP-reduce, set $d=2$: $G(2;t)=0 \Rightarrow$ new linear relation among the $E_n(2;t)$

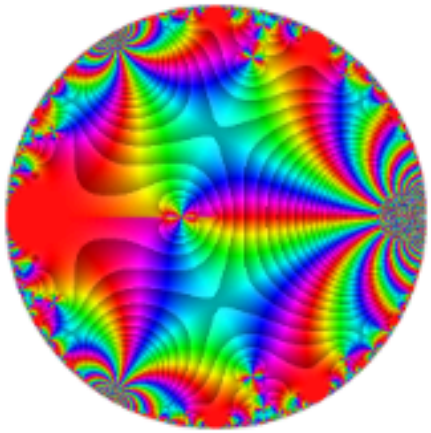
- Removes elliptic functions from the $1/\epsilon^2$ pole; its $O(\epsilon)$ extension does the same for $1/\epsilon \Rightarrow$ renormalizability restored
- Effectively eliminates $E_6(2;t)$: not implied by IBP, which holds for generic d
- Novel here; first seen in the 2-loop sunset with 3 distinct masses [Remiddi & Tancredi '14]
  All relations checked numerically with pySecDec

Evaluating the elliptics

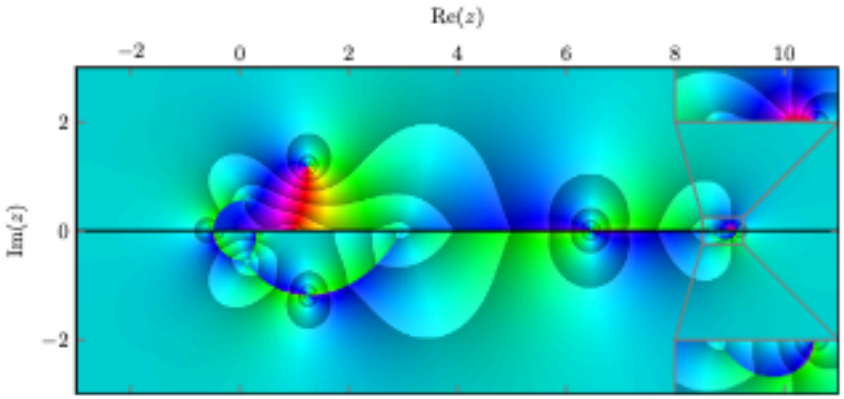
- A wonderland of pretty pictures: holomorphic maps & domain-coloring plots, e.g.



$t(q)$



$E_3(2;t) - E_3(2;0)$



$\rho_F(z) - F_1(1)$

- A nightmare of complex analysis, branch cuts, differential equations, integration, . . .

Results

Three-loop order

N^{3}LO (three loops) [JHEP02(2026)154], up to terms $\propto L_\pi$:

$$\begin{aligned}\bar{\Pi}_T^{\mathrm{N^3LO}}(t) = {} & + \left[E_{1,2,3}(2;t),\, \bar{E}_{5,6}(4;t),\, J^{(n)}(t),\, \zeta(3),\, \pi^2\right] \\ & + t\,B(t)^2\left[(l_2 - 2l_1 - 2l_6)\,t + 2l_4\right] \\ & - 2t\,B(t)\,l_6\,(4l_4 - l_6 t) \\ & + 4t\,B(t)\,(r_{V1} + r_{V2}\,t) \\ & - 16t\,l_4\,c_{56} \\ & + 16t\,(\tilde{c}_{332} - \tilde{c}_{333}) - 8t^2\,\tilde{c}_{459}\end{aligned}$$

function of t only | LECs known | LECs to be determined | LECs unknown

dispersive & FVE | FVE only (?) | neither

Bubble function & pion vector form factor NNLO LECs

$$B(t) := \frac{4-t}{3t}\,J^{(1)}(t) - \frac{1+3L_\pi}{9}, \qquad J^{(1)}(t) := \int_0^1 dx\,\log\left[1 - x(1-x)t\right]$$

$$L_\pi := \log(M_\pi^2/\mu^2), \qquad r_{V1} := -4\,(2c_{53} + c_{35} + 4c_6)\,, \qquad r_{V2} := 4\,(c_{53} - c_{51})$$

[JHEP02(2026)154; Bijnens, Colangelo & Talavera '98]

[. . .]: combination, with coefficients rational in t, of $E_{1,2,3}(2;t)$ (three-loop sunsets in $d = 2$) and $\bar{E}_{5,6}(4;t)$ (finite parts of the cat's eyes in $d = 4 - 2\epsilon$), entering linearly; polynomials in $J^{(1,2,3)}(t)$; and the constants $\zeta(3)$, π^2 [JHEP02(2026)154, eqs. (5.5)–(5.7)]

The LECs r_{V1} and r_{V2}

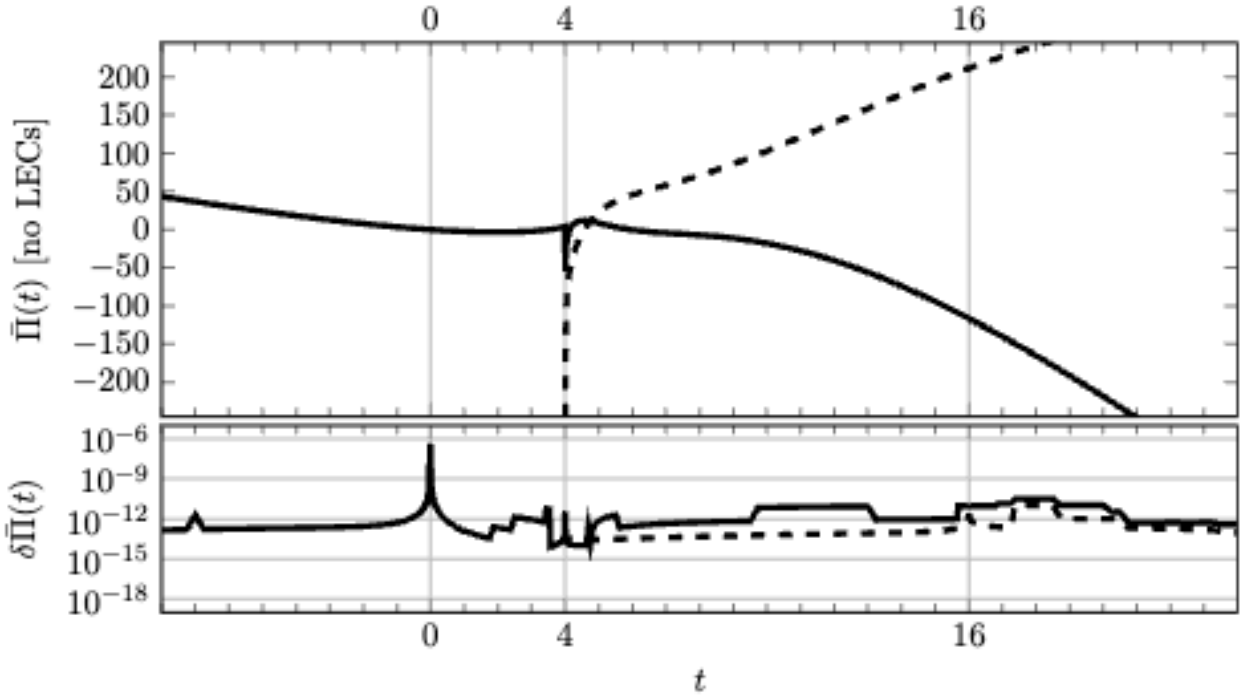
Pion vector form factor and charge radius (schematically)

$$F_V^\pi(s) = 1 + \tfrac{1}{6}\langle r^2\rangle_V^\pi\, s + c_V^\pi\, s^2 + O(s^3)$$
$$M_\pi^2\, \langle r^2\rangle_V^\pi = \xi\,[l_6] + \xi^2\big([l_1, l_2, l_4, l_6] + 6\, r_{V1}\big) + O(\xi^3)$$
$$M_\pi^4\, c_V^\pi = \frac{\xi}{60} + \xi^2\big([l_1, l_2, l_4, l_6] + r_{V2}\big) + O(\xi^3)$$

- Vector meson dominance: $r_{V1} \approx -6.2$, $r_{V2} \approx +6.5$ [Bijnens, Colangelo & Talavera '98]
- Data: $r_{V2} = 4.0 \pm 1.2$ [Bijnens & Ecker '14]; naive extraction from the charge radius: $r_{V1} = -5 \pm 8$ [JHEP02(2026)154]
- Lattice QCD: potentially better precision (efforts under way), and access to the M_π dependence [cf. Colangelo et al. '22]

LECs in the normalization of JHEP02(2026)154: $l_i^r = \pi_{16}\, l_i^q$, $r_{Vi}^r = \pi_{16}^2\, r_{Vi}^q$

The LEC-independent part



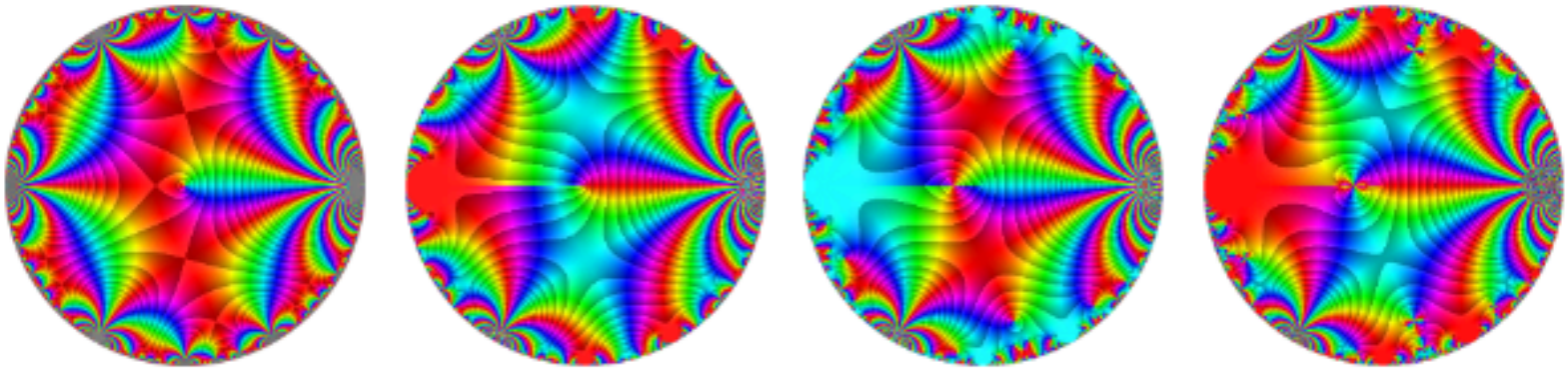
LEC-independent part of $\bar{\Pi}_T^{\mathrm{N^3LO}}(t)$: real (solid) and imaginary (dashed) parts; below: numerical uncertainty

Evaluation of the master integrals [JHEP08(2026)116]

- Stable over the entire complex t plane
- 10^3–10^{12} times more precise than pySecDec [Borowka et al. '18]
- $E_{1,2,3}$, $\bar{E}_4$: much faster than pySecDec; $\bar{E}_{5,6}$: comparable

Summary and outlook

- Three-loop HVP in two-flavor χPT: amplitude computed, renormalized and published [JHEP02(2026)154]
- Elliptic master integrals: fast, precise evaluation for all complex t [JHEP08(2026)116]; code: github.com/mssjo/HVP-numerics
- First phenomenological applications underway
- Beginning to examine finite-volume effects at N^{3}LO



JHEP 02 (2026) 154 [arXiv:2510.12885] JHEP 08 (2026) 116 [arXiv:2603.15252]