The Collins-Soper kernel from a vacuum soft function

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with
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Lattice field theory at the frontier of nuclear and particle physics
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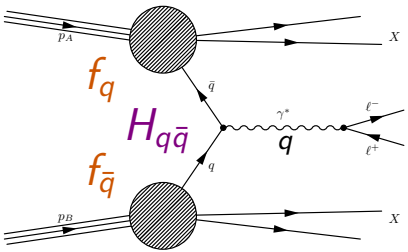
TMD factorization in Drell-Yan scattering



- $\Lambda_{\rm QCD} \lesssim |\vec{q}_\perp| \ll Q = \sqrt{q^2}$
- Rapidity divergences in TMDPDFs need to be regulated
- Dependence on the Collins-Soper scale $\zeta_{a,b}$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}Q\mathrm{d}Y\mathrm{d}^2\vec{q}_\perp} = \sum_q H_{q\bar{q}}\left(Q,\mu\right) \int \mathrm{d}^2\vec{b}_\perp e^{i\vec{b}_\perp\cdot\vec{q}_\perp} f_q\left(x_a,\vec{b}_\perp,\mu,\zeta_a\right) f_{\bar{q}}\left(x_b,\vec{b}_\perp,\mu,\zeta_b\right)$$

$$\times\left[1+\mathcal{O}\left(\frac{q_\perp^2}{Q^2},\frac{\Lambda_{\rm QCD}^2}{Q^2}\right)\right] \qquad \zeta_a\zeta_b = Q^4$$

$\zeta_a = 2(x_a P_A^+ e^{-y_n})^2$, $\quad \zeta_b = 2(x_b P_B^- e^{+y_n})^2$, $\quad y_n$: arbitrary rapidity parameter

$$f_q\left(x_a, \vec{b}_\perp, \mu, \zeta_a\right) = \lim_{\substack{\epsilon\to 0 \\ y_B\to-\infty}} Z_{\rm UV}\left(\mu, \zeta_a, \epsilon\right) \frac{f_q^{\rm bare}\left(x_a, \vec{b}_\perp, \epsilon, y_B, x_a P^+\right)}{\sqrt{S_q\left(b_\perp, \epsilon, 2y_n - 2y_B\right)}}$$

UV scale evolution:

$$\gamma_\mu^q(\mu,\zeta) = \frac{{\rm d}\log f_q\left(x, \vec{b}_\perp, \mu, \zeta\right)}{{\rm d}\log\mu}$$

Rapidity evolution (CS kernel):

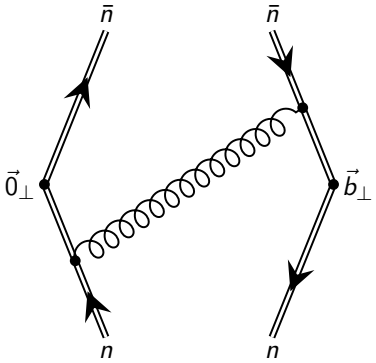
$$\gamma_q(b_\perp,\mu) = \frac{{\rm d}\log f_q\left(x, \vec{b}_\perp, \mu, \zeta\right)}{{\rm d}\log\sqrt{\zeta}}$$

The CS kernel controls the rapidity dependence of the soft function:

$$S_q\left(b_\perp, y_A, y_B, \mu\right) = S_I\left(b_\perp, \mu\right) e^{\gamma_q(b_\perp,\mu)(y_A - y_B)} \left(1 + \mathcal{O}\left(e^{-2(y_A-y_B)}\right)\right)$$

$y_{A,B}$: rapidities of the Wilson-line directions; S_I: intrinsic soft function

Rapidity divergence



one-gluon exchange between the Wilson lines

One-loop soft function, lightlike Wilson lines:

$$\int \frac{\mathrm{d}k^+\mathrm{d}k^-}{(2\pi)^2} \frac{1}{k^+k^- - k_\perp^2 + i0} \frac{1}{n\cdot k - i0} \frac{1}{\bar{n}\cdot k + i0}$$

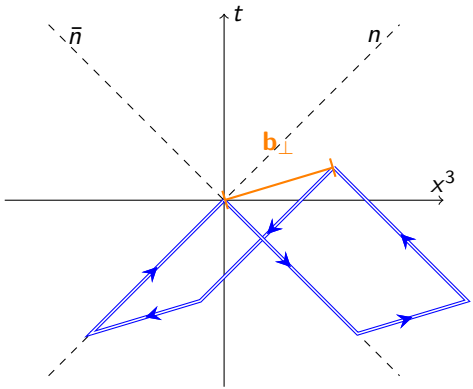
$$= \int \frac{\mathrm{d}\alpha}{(2\pi)^2} \frac{1}{\alpha - k_\perp^2 + i0} \frac{1}{\alpha - i0} \int_{-\infty}^{\infty} \mathrm{d}y$$

$$\alpha = k^+k^-, \qquad y = \frac{1}{2}\ln\left(\frac{k^-}{k^+}\right)$$

$$k^\pm = k^0 \pm k^3, \quad k^- = n\cdot k, \quad k^+ = \bar{n}\cdot k$$

Integrand independent of the gluon rapidity y: divergence from $y \to \pm\infty$

Regulating the soft function

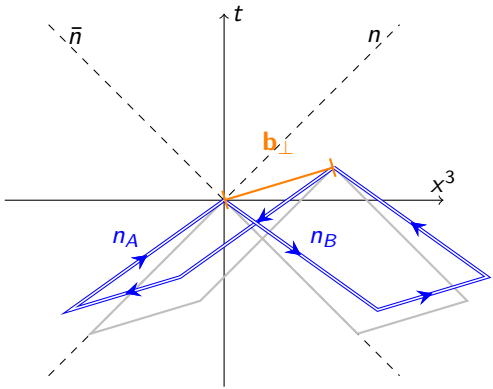


Soft function defined on the lightcone:

$$n^2 = \bar{n}^2 = 0$$

$$n = (1, 0, 0, 1),$$
$$\bar{n} = (1, 0, 0, -1)$$

Regulating the soft function



Soft function defined on the lightcone:

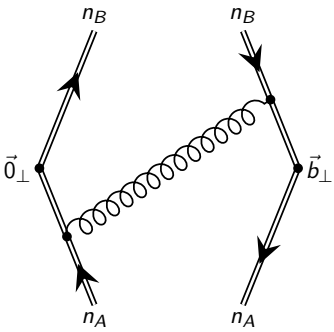
$$n^2 = \bar{n}^2 = 0$$

$$n = (1, 0, 0, 1),$$
$$\bar{n} = (1, 0, 0, -1)$$

Regulate rapidity divergence with spacelike Wilson lines:

$$n_A \equiv n - e^{-2y_A}\bar{n},$$
$$n_B \equiv \bar{n} - e^{+2y_B}n,$$
$$y_A, -y_B \to \infty$$

Rapidity divergence off the lightcone



Wilson lines along spacelike n_A, n_B

One-loop soft function, spacelike Wilson lines:

$$\int \frac{\mathrm{d}k^+\mathrm{d}k^-}{(2\pi)^2} \frac{1}{k^+k^- - k_\perp^2 + i0} \frac{1}{n_A\cdot k} \frac{1}{n_B\cdot k}$$
$$= \int \frac{\mathrm{d}\alpha}{(2\pi)^2} \frac{1}{\alpha - k_\perp^2 + i0} \frac{1}{\alpha - i0} \int_{-\infty}^{\infty} \mathrm{d}y\, F(y)$$

$$F(y) = \frac{1}{\big(1 - e^{-2(y+y_A)}\big)\big(1 - e^{2(y+y_B)}\big)}$$

$n_A\cdot k = k^- - e^{-2y_A}k^+, \quad n_B\cdot k = k^+ - e^{+2y_B}k^-$

$$\int_{-\infty}^{\infty} \mathrm{d}y\, F(y) = (y_A - y_B)\Big(1 + \mathcal{O}\big(e^{-2(y_A - y_B)}\big)\Big)$$

Rapidity divergence $\to y_A - y_B$, exponentiates to $S_q \propto e^{\gamma_q (y_A - y_B)}$

In the LaMET (Large Momentum Effective Theory) framework:

$$\sqrt{S_I\left(b_\perp,\mu\right)}\,\tilde{f}_\Gamma\left(x,b_\perp,P^z,\mu\right)$$
$$= f\left(x,b_\perp,\mu,\zeta\right) H_f\left(x,P^z,\mu\right) \exp\left[\frac{1}{2}\log\frac{(2xP^z)^2}{\zeta}\gamma_q\left(b_\perp,\mu\right)\right]$$
$$+\mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{xP^z},\frac{1}{b_\perp(xP^z)}\right)$$

[Ebert, *et al.*, 2019], [Ji, Liu, Liu, 2019], [Ebert, *et al.*, 2022]

$\tilde{f}_\Gamma$: quasi-TMD beam function, H_f: perturbative hard kernel

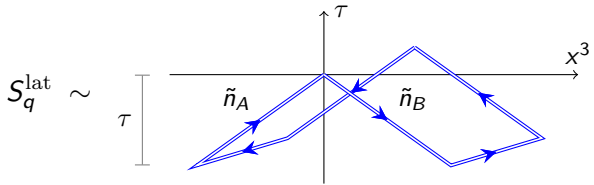
S_I: intrinsic soft function [Ji, Liu, Liu, 2020]

This talk: γ_q from a **vacuum** soft function, with no hadron states

Motivated by a proposal by Liu in [Ji, Liu, Liu, 2019]

Euclidean space directional vectors with purely imaginary time components

$$\tilde{n}_A = \left(in_A^0, 0, 0, n_A^3\right), \quad \tilde{n}_B = \left(in_B^0, 0, 0, n_B^3\right)$$



$$r_a = \frac{n_A^3}{n_A^0} = \frac{1+e^{-2y_A}}{1-e^{-2y_A}}, \quad r_b = \frac{n_B^3}{n_B^0} = \frac{1+e^{2y_B}}{1-e^{2y_B}}$$

Auxiliary field definition of the Wilson line

Write the Wilson line in terms of one-dimensional 'fermions' that live along the path:

$$P\exp\left\{-ig\int_{s_i}^{s_f}\mathrm{d}s n^\mu A_\mu(y(s))\right\}$$
$$= Z_\psi^{-1}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\psi(s_f)\bar{\psi}(s_i)\exp\left\{i\int_{s_i}^{s_f}\mathrm{d}s\left(\bar{\psi}i\partial_s\psi - g\bar{\psi}n\cdot A\psi\right)\right\}$$

[Gervais, Neveu 1980], [Aref'eva 1980]

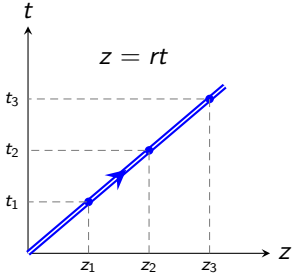
Auxiliary field propagator:

$$in\cdot DH_n(y) = \delta(y) \overset{\text{Euclidean space}}{\longrightarrow} -i\tilde{n}\cdot D_E H_r(y) = \delta(y), \qquad \tilde{n} = (in^0, \vec{n})$$

Meaningful solution only obtained with a UV cutoff, since the auxiliary field Hamiltonian is unbounded from below. [Aglietti, *et al.* 1992], [Aglietti, 1994]

Interpretation of the path variable in Euclidean space

Minkowski: $\mathrm{d}x^\mu(s) = n^\mu \mathrm{d}s$



$t_1 < t_2 < t_3 \;\Leftrightarrow\; z_1 < z_2 < z_3$

Interpretation of the path variable in Euclidean space

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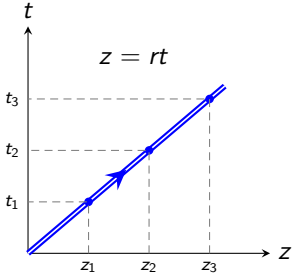
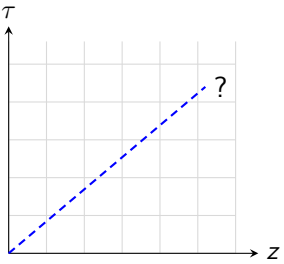


Euclidean: $\tilde{n} = (in^0, \vec{n})$



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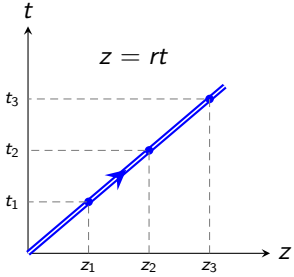

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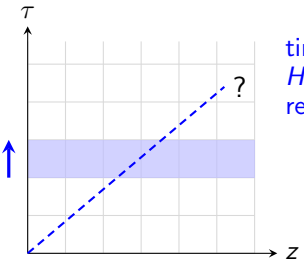

time-ordered:
$H(\tau+a,\cdot) \leftarrow H(\tau,\cdot)$
requires $|r| < 1$

$t_1 < t_2 < t_3 \;\Leftrightarrow\; z_1 < z_2 < z_3$

Interpretation of the path variable in Euclidean space

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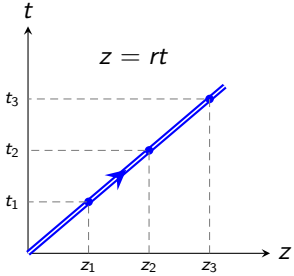

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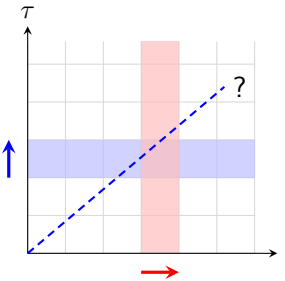


time-ordered:
$H(\tau{+}a, \cdot) \leftarrow H(\tau, \cdot)$
requires $|r| < 1$

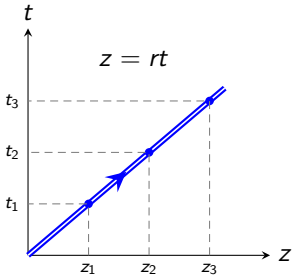
space-ordered:
$H(\cdot, z{+}a) \leftarrow H(\cdot, z)$
requires $|r| > 1$

$t_1 < t_2 < t_3 \;\Leftrightarrow\; z_1 < z_2 < z_3$

t-order $\neq$ z-order: pick one

Interpretation of the path variable in Euclidean space

Minkowski: $\mathrm{d}x^\mu(s) = n^\mu \mathrm{d}s$



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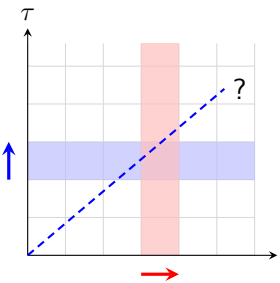


time-ordered:
$H(\tau+a,\cdot) \leftarrow H(\tau,\cdot)$
requires $|r| < 1$

space-ordered:
$H(\cdot, z+a) \leftarrow H(\cdot, z)$
requires $|r| > 1$

$t_1 < t_2 < t_3 \Leftrightarrow z_1 < z_2 < z_3$ t-order $\neq$ z-order: **we use z-ordering**

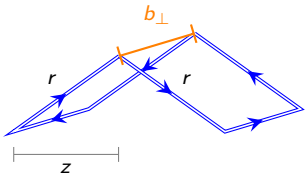
Perturbative computation: t-ordering needs $|r| < 1$, z-ordering needs $|r| > 1$
in order to suppress exponential growth from unbounded Hamiltonian
$\Rightarrow$ z-ordered evolution with $|r| > 1$, spacelike regulator in Minkowski space

What we want to measure on the lattice

Wilson lines of length z along $\tilde{n}$, built from z-ordered auxiliary field propagators

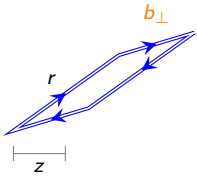
Unsubtracted soft function:

$$S_Q^{\text{lat}}(b_\perp, z; r, r; a) \sim$$



Rectangular loop:

$$S_{\text{rect}}^{\text{lat}}(b_\perp, z; r; a) \sim$$



$S_{\text{rect}}^{\text{lat}}(b_\perp, z; -r; a)$: mirror image (other leg)

Cancellation of self-energy and cutoff effects via

$$S_q^{\text{lat}} \sim \frac{S_Q^{\text{lat}}(b_\perp, z; r, r; a)}{\sqrt{S_{\text{rect}}^{\text{lat}}(b_\perp, z; r; a)\, S_{\text{rect}}^{\text{lat}}(b_\perp, z; -r; a)}}$$

Lattice setup

Ensemble

- quenched, $32^3 \times 64$, $\beta = 6.30168$, $a = 0.06$ fm [Detmold, Endres 2018]
- 131072 sources per configuration
- 450 configurations

Measurements

- $b_\perp \in \{3, 5, 7, 9\}$ $a = 0.18$–0.54 fm
- $z \in \{1, \ldots, 6\}$ $a = 0.06$–0.36 fm
- $1/r = 0.99$

Analysis

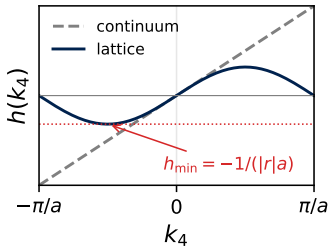
- Bootstrap errors, 1000 resamples over configurations

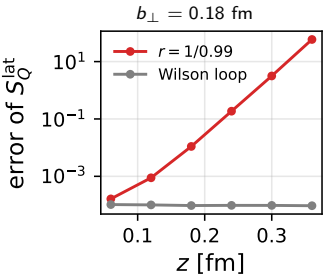
Unbounded "Hamiltonian" on the lattice

Lattice auxiliary field action, evolved along z:

$$\left(\Delta_z^- - \tfrac{i}{r}\Delta_t\right) H_r(t,z) = \delta_{t,t_0}\delta_{z,z_0}$$

$$H_r(k_4,z) \sim e^{-h(k_4)\,z}, \qquad h(k_4) = \frac{k_4}{r} \overset{\text{lattice}}{\longrightarrow} \frac{1}{ra}\sin(k_4 a)$$





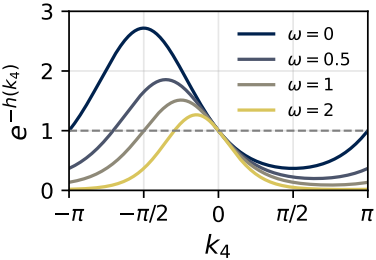
Wilson term for the auxiliary field

Add a Wilson term, ω, to the auxiliary field action:

$$\left(\Delta_z^- - \tfrac{i}{r}\Delta_t - \tfrac{1}{|r|}\tfrac{\omega}{2}\Delta_t^+\Delta_t^-\right)H_r(t,z) = \delta_{t,t_0}\delta_{z,z_0}$$

$$h(k_4) = \tfrac{1}{r}\sin k_4 + \tfrac{1}{|r|}\,\omega\big(1-\cos k_4\big)$$

- $\mathcal{O}(a)$ Wilson term
- Raises the bottom of the spectrum: $h_{\text{min}} = \frac{1}{|r|}\big(\omega - \sqrt{1+\omega^2}\big)$
- $\omega \to 0$ recovers the original action



free field, $r = 1$

As in NRQCD, with $n = 3$: [Lepage, *et al.* 1992]

$$H_r(t,z+1) = \left(1-\tfrac{\mathcal{H}}{2n}\right)^n U_z^\dagger(t,z)\left(1-\tfrac{\mathcal{H}}{2n}\right)^n H_r(t,z), \qquad \mathcal{H} = -\tfrac{i}{r}\Delta_t - \tfrac{1}{|r|}\tfrac{\omega}{2}\Delta_t^+\Delta_t^-$$

Free-field "transfer matrix":

$$T(k_4) = \left(1 - \frac{h(k_4)}{2n}\right)^{2n} = e^{-h(k_4)}\Big(1 + \mathcal{O}(a)\Big)$$

Extremes of $h(k_4) = \frac{1}{r}\sin k_4 + \frac{1}{|r|}\,\omega(1 - \cos k_4)$:

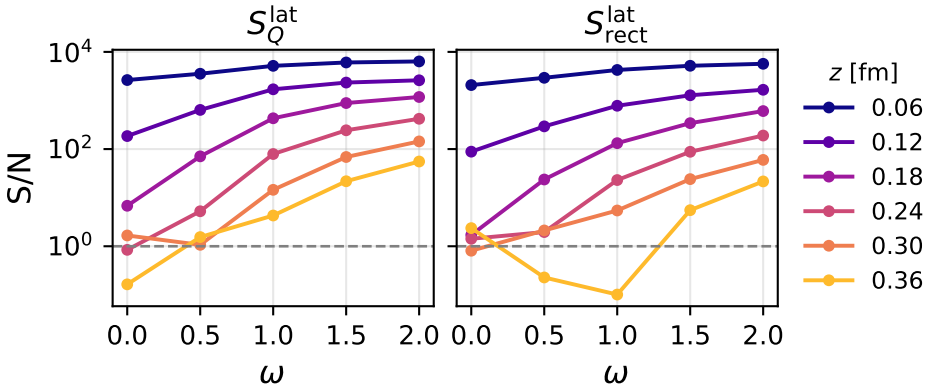
$$h_{\min} = \tfrac{1}{|r|}\big(\omega - \sqrt{1+\omega^2}\big) < 0, \qquad h_{\max} = \tfrac{1}{|r|}\big(\omega + \sqrt{1+\omega^2}\big)$$

condition on $h_{\max}$, $\left|1 - \frac{h_{\max}}{2n}\right| < 1 \;\Leftrightarrow\; h_{\max} < 4n$

$$\omega < \frac{(4n|r|)^2 - 1}{8n|r|} \;=\; 5.96 \quad (n = 3,\ |r| = 1)$$

we use $\omega \in \{0, 0.5, 1, 1.5, 2\}$

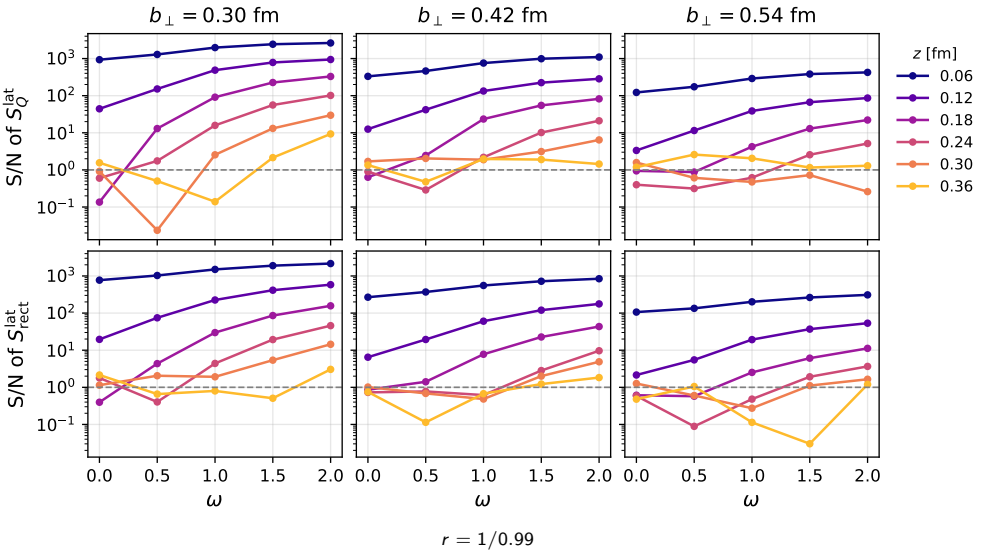
Effect of Wilson term on the noise



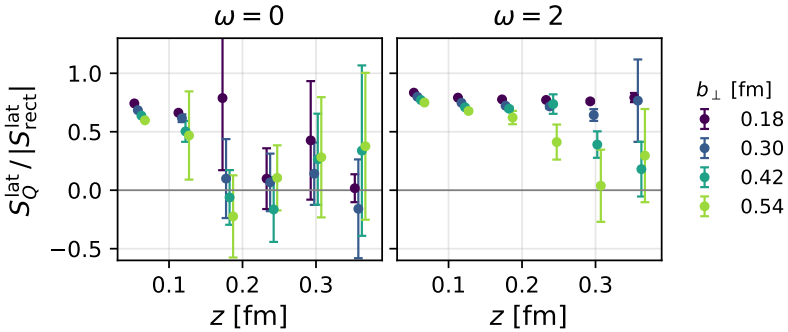
$b_\perp = 0.18$ fm, $r = 1/0.99$; $32^3 \times 64$, $\beta = 6.30168$, $a = 0.06$ fm, 131072 sources $\times$ 450 configs (418 for $S_{\text{rect}}^{\text{lat}}$, $\omega = 0$)

S/N at $z = 0.24$ fm, $\omega = 0 \to 2$: S_Q^{lat}: $0.8 \to 420$, $S_{\text{rect}}^{\text{lat}}$: $1.4 \to 190$

S/N against ω at larger $b_\perp$



$$S_q^{\text{lat}} \sim \frac{S_Q^{\text{lat}}(b_\perp, z; r, r; a)}{\left|S_{\text{rect}}^{\text{lat}}(b_\perp, z; r; a)\right|}, \qquad S_{\text{rect}}^{\text{lat}}(b_\perp, z; -r; a) = S_{\text{rect}}^{\text{lat}}(b_\perp, z; r; a)^*$$



$r = 1/0.99$; 418 ($\omega = 0$) and 450 ($\omega = 2$) configs

Preliminary: the ratio depends on $\omega \Rightarrow \omega \to 0$ / continuum extrapolation needed

Conclusion and outlook

- Euclidean space soft function with complex directional vectors maps directly to Minkowski space result
- Wilson term on the auxiliary field suppresses the exponential noise growth: $e^{z/(|r|a)} \to e^{0.24\, z/(|r|a)}$ at $\omega = 2$
- $S_Q^{\rm lat}/|S_{\rm rect}^{\rm lat}|$ is now precise out to $z \approx 0.36$ fm at small $b_\perp$

Next steps:

- $\omega \to 0$ and continuum extrapolation of the soft function
- More configurations for large $b_\perp$
- Further improvements to noise with APE smearing
- Fit rapidity dependence to obtain intrinsic soft function, S_I, and CS kernel, γ_q