Quantum Many-body Scars as Protected Subgraph in the Fock Space Lattice: A Study in 2D Constraint System

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I. Introduction

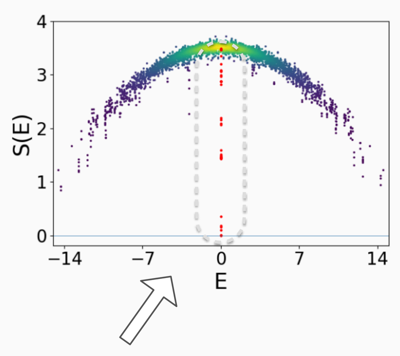
Understanding the fundamental theory of preventing thermalization is crucial for practical quantum device development.

Thermalization in isolated quantum systems differs from classical systems due to *unitary* time evolution, where wave function probability amplitudes stay constant.

$$|\psi(0)\rangle = \sum_\alpha c_\alpha |E_\alpha\rangle \xrightarrow{t\to\infty} |\psi(t)\rangle = \sum_\alpha c_\alpha e^{-iE_\alpha t}|E_\alpha\rangle$$

Instead, we observe physical observables (e.g. energy) over time to assess whether they converge to the microcanonical ensemble.

$$\langle \hat{O}\rangle_\infty = \lim_{T\to\infty}\frac{1}{T}\int_0^T dt \langle\psi(t)|\hat{O}|\psi(t)\rangle \to \langle\hat{O}\rangle_{mc} \quad ?$$



Partial Answer: Eigenstate Thermalization Hypothesis (ETH)

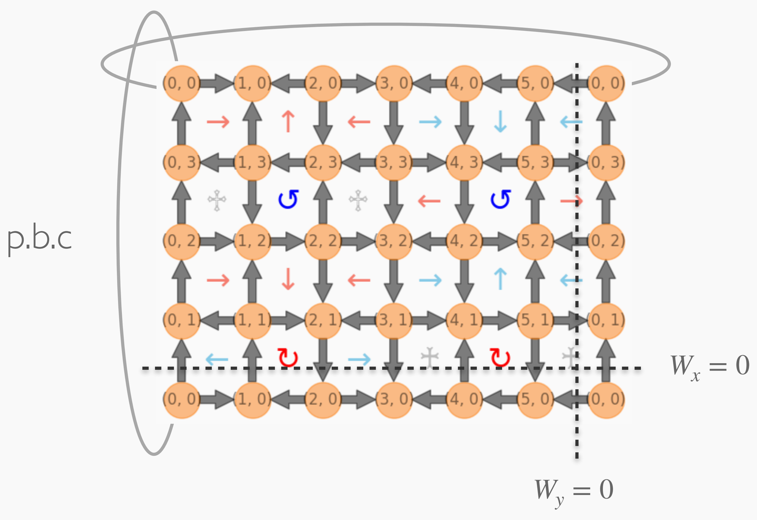
$$\langle E_\alpha|\hat{O}|E_\beta\rangle = \langle\hat{O}\rangle_{mc}(\bar{E})\,\delta_{\alpha\beta} + f(\omega,\bar{E})\,e^{-S^{\dagger}_{th}/2}R_{\alpha\beta}$$

For operators satisfying this ansatz, off-diagonal elements average out over time, resulting in thermalization.

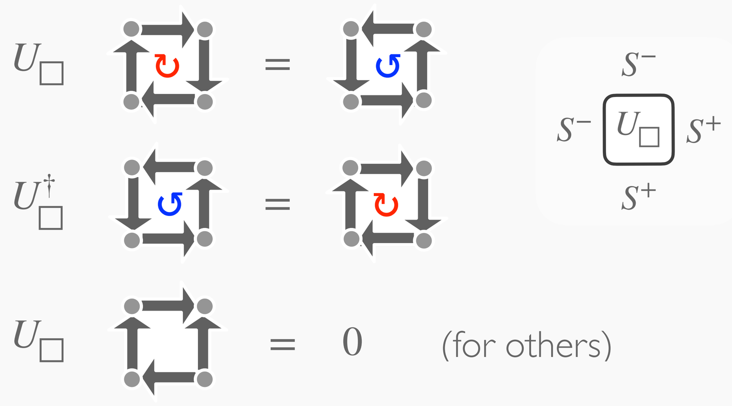
Scars exemplify ETH violations as excited states with area-law entanglement entropy. Many-body localization (MBL) serves as another example.

II. U(1) Lattice Gauge Models

Each link has spin-1/2 *up* (→, ↑) or *down* (←, ↓) as electric flux.



Closed loops having orientation can be flipped by $U_\square$ or $U_\square^\dagger$.

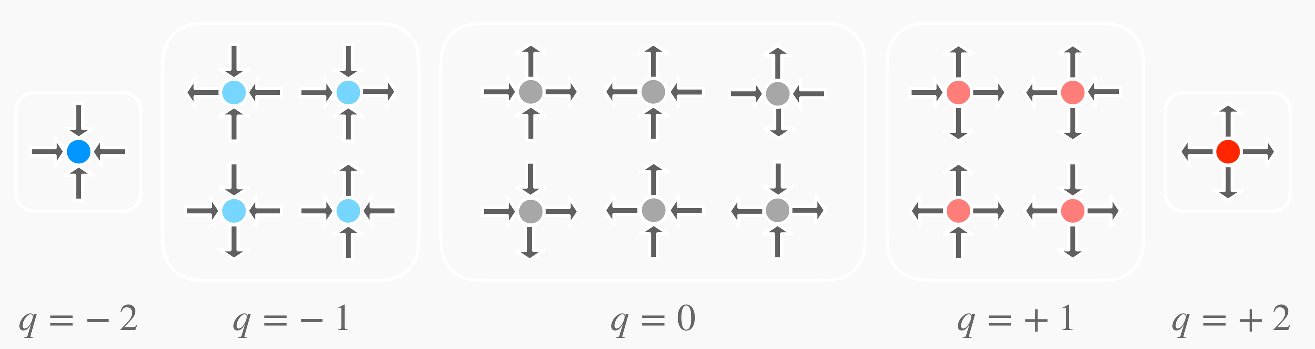


- *Flippable* plaquettes: ↻ and ↺
- Nearly-*flippable*: pink/aqua arrows
- Global flux symmetry $W_\mu = \frac{1}{L_\mu}\sum_r E_{r,\hat{\mu}}$

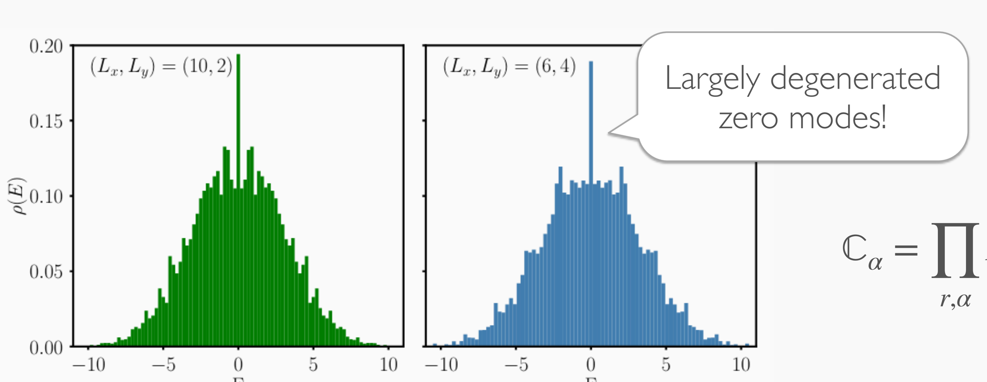
$$H = -J\sum_\square (U_\square + U_\square^\dagger) + \lambda\sum_\square (U_\square + U_\square^\dagger)^2$$

Flip flippables | Count flippables

Gauss Laws (gauge constraints) are required on each lattice site

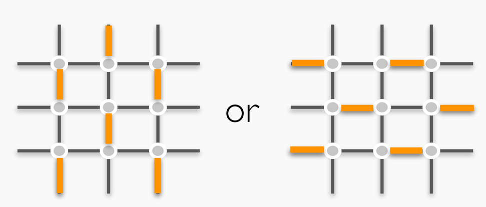


Index Theorem [2] - spectral reflection symmetry



$$\{\hat{O}_{kin}, \mathbb{C}_\alpha\} = 0$$

$$\mathbb{C}_\alpha = \prod_{r,\alpha} E_{r,\alpha} =$$

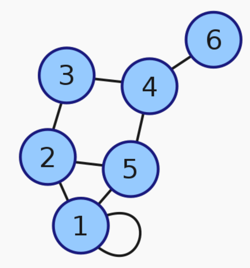


Classification of Scars [1]	• Type I	• Type II	• Type IIIA	IIIB, IIIC ($\lambda = 0$)
	$\langle\hat{O}_{kin}\rangle = 0$	$\langle\hat{O}_{kin}\rangle = 0$	$\langle\hat{O}_{kin}\rangle = \mathbb{Z}$	$\langle\hat{O}_{kin}\rangle = \mathbb{Z}$ or $\mathbb{R}\backslash\mathbb{Q}$
$\lambda \neq 0$	$\langle\hat{O}_{pot}\rangle = \mathbb{Z}$	$\langle\hat{O}_{pot}\rangle = \overline{\{\mathbb{Z}\}} = \mathbb{Z}$	$\langle\hat{O}_{pot}\rangle = \mathbb{Z}$	$\langle\hat{O}_{pot}\rangle$ = N/A

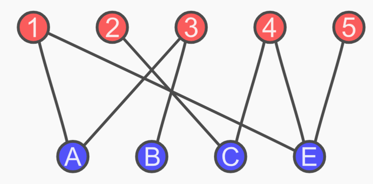
III. Graph Theory

- Adjacency Matrix

$$A = \begin{pmatrix} 2&1&0&0&1&0\\ 1&0&1&0&1&0\\ 0&1&0&1&0&0\\ 0&0&1&0&1&1\\ 1&1&0&1&0&0\\ 0&0&0&1&0&0 \end{pmatrix}$$



- Bipartite Graph



The spectrum of the bipartite graph is spectral reflectional symmetric.

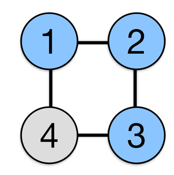
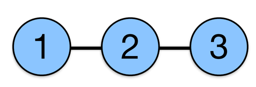
Cauchy Interlacing Theorem [3]

$$A = \begin{pmatrix} B & \cdots \\ \vdots & \ddots \end{pmatrix} \quad \text{Spectrum of} \quad \begin{matrix} A: \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \\ B: \mu_1 \le \mu_2 \le \cdots \le \mu_m \end{matrix} \quad \text{with} \quad m < n$$

$$\text{then} \quad \lambda_i \le \mu_i \le \lambda_{n-m+i} \quad \text{for} \quad i = 1, \cdots, m$$

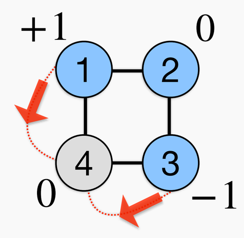
Meaning in graph: Let B be a vertex-deleting subgraph of A, what conditions allow the induced subgraph B to share the same eigenpairs with A?

- ***Example:***

	Graph	Spectrum	Eigenvectors
Cycle graph $C_4 =$		$\{-2, 0, 0, 2\}$	$\psi_{E=0} \sim (0\ \ 1\ \ 0\ \ -1)^T$ $\psi_{E=0} \sim (1\ \ 0\ \ -1\ \ 0)^T$
Path graph $P_3 =$		$\{-\sqrt{2}, 0, \sqrt{2}\}$	$\psi_{E=0} \sim (1\ \ 0\ \ -1)^T$

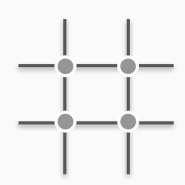
Perfect destructive interference on the graph

The scars turn out to be the more complicated version of this example, they are eigenstates from the subgraph.



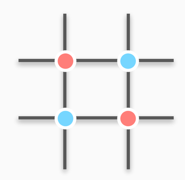
IV. Numerical Methods

- Quantum Link Model (QLM)

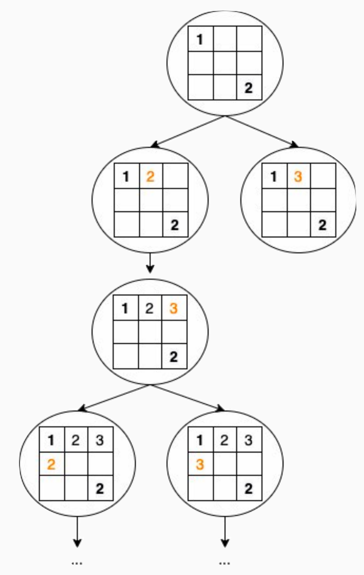


Hilbert space in Quantum Link Model			
(L_x, L_y)	Gauss law	$(W_x, W_y) = (0,0)$	$(k_x, k_y) = (0,0)$
(8, 2)	7074	2214	142
(10, 2)	61098	17906	902
(12, 2)	539634	147578	6166
(14, 2)	4815738	1232454	44046
(16, 2)	43177794	10393254	324862
(4, 4)	2970	990	70
(6, 4)	98466	32810	1384
(8, 4)	3500970	1159166	36360
(6, 6)	16448400	5482716	152416

- Quantum Dimer Model (QDM)



Hilbert space in Quantum Dimer Model			
(L_x, L_y)	Gauss law	$(W_x, W_y) = (0,0)$	$(k_x, k_y) = (0,0)$
(8, 2)	1156	384	29
(10, 2)	6728	2004	106
(12, 2)	39204	10672	460
(14, 2)	228488	57628	2077
(6, 4)	3108	1456	71
(8, 4)	39952	17412	571
(10, 4)	537636	216016	5490
(12, 4)	7379216	2739588	57379
(6, 6)	90176	44176	1256
(8, 6)	3113860	1504896	31464

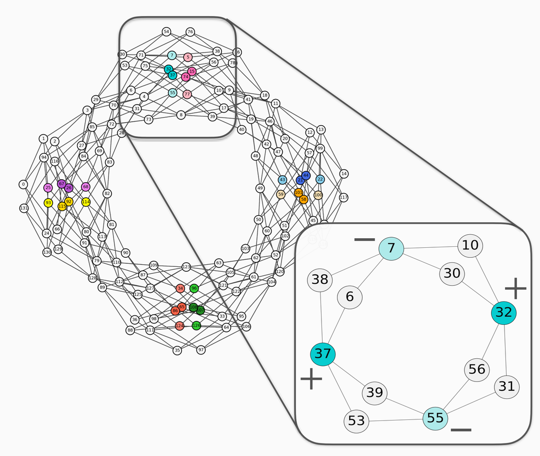


Enumerate all possible basis with deep-first search (DFS).

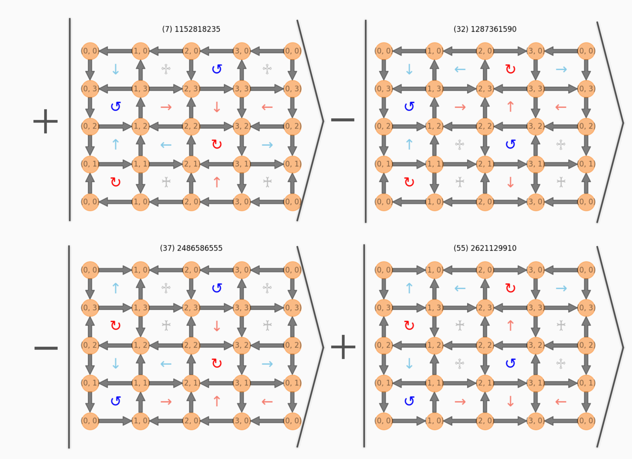
We develop an efficient algorithm based on the idea of subgraph and the destructive interference to identify all Type-I scars found in [1], originally detected through Exact Diagonalization (ED).

V. Results

- One of the Type-I scars in $\hat{O}_{kin}$ as a graph (QDM 4x4)



- The basis of this Type-I scar (QDM 4x4)



The example shows a Type-I scar formed by 4 computational basis with specific sign arrangements, resulting in their cancellation within the graph.

We observe non-thermalizing excited states, especially Type-I scars, resulting from extensive cancellation within the Hilbert space. These scars represent embedded subgraphs where cancellation occurs at the boundary, explaining their resistance to thermalization.

Type-II & Type-III scars entail more complex cancellation that remain not fully understood, thus leaving as our future research.

1. S. Biwas, D. Banerjee, and A. Sen, SciPost Phys. 12, 148 (2022).
2. M. Schecter and T. Iadecola, Phys. Rev. B 98, 035139 (2018).
3. W. H. Haemers, Linear Algebra and its Applications, 226-228, 593 (1995).

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Scan for the Github repo